Children's Thinking at the Upper Levels of an Area Learning Trajectory

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WHERE DISCOVERIES BEGIN

Describing the LT for Area Measurement Beyond the Emergence of Units and Unit Iteration

- By Grade 2, the children in the Northeast cohort (from pre-K to Gr 2) exhibited thinking at initial composite structuring at times (ICS), and clearly established units of area to help them measure the area of a region.

- Area Quantity Recognizer
- Physical Coverer and Counter
- Complete Coverer and Counter
- Area Unit Relater and Repeater
- Initial Composite Structurer
- Area Row and Column Structurer
- Array Structurer

Increasing Sophistication

Describing the LT for Area Measurement Beyond the Emergence of Units and Unit Iteration

- By Grade 2, the children in the Northeast cohort (from pre-K to Gr 2) exhibited thinking at initial composite structuring at times (ICS), and clearly established units of area to help them measure the area of a region.

- The children in the Midwest cohort (from Gr 2 to Gr 5) exhibited some early use of tiles and units without grouping clearly (CCC and AURR) during Grade 2.

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Increasing Sophistication

Describing the LT for Area Measurement Beyond the Emergence of Units and Unit Iteration

- The children in the Midwest cohort (from Gr 2 to Gr 5) either reached or established the use of arrays as mental objects to examine regions using length and width reports along edges of figures by Grade 5.

- Area Quantity Recognizer
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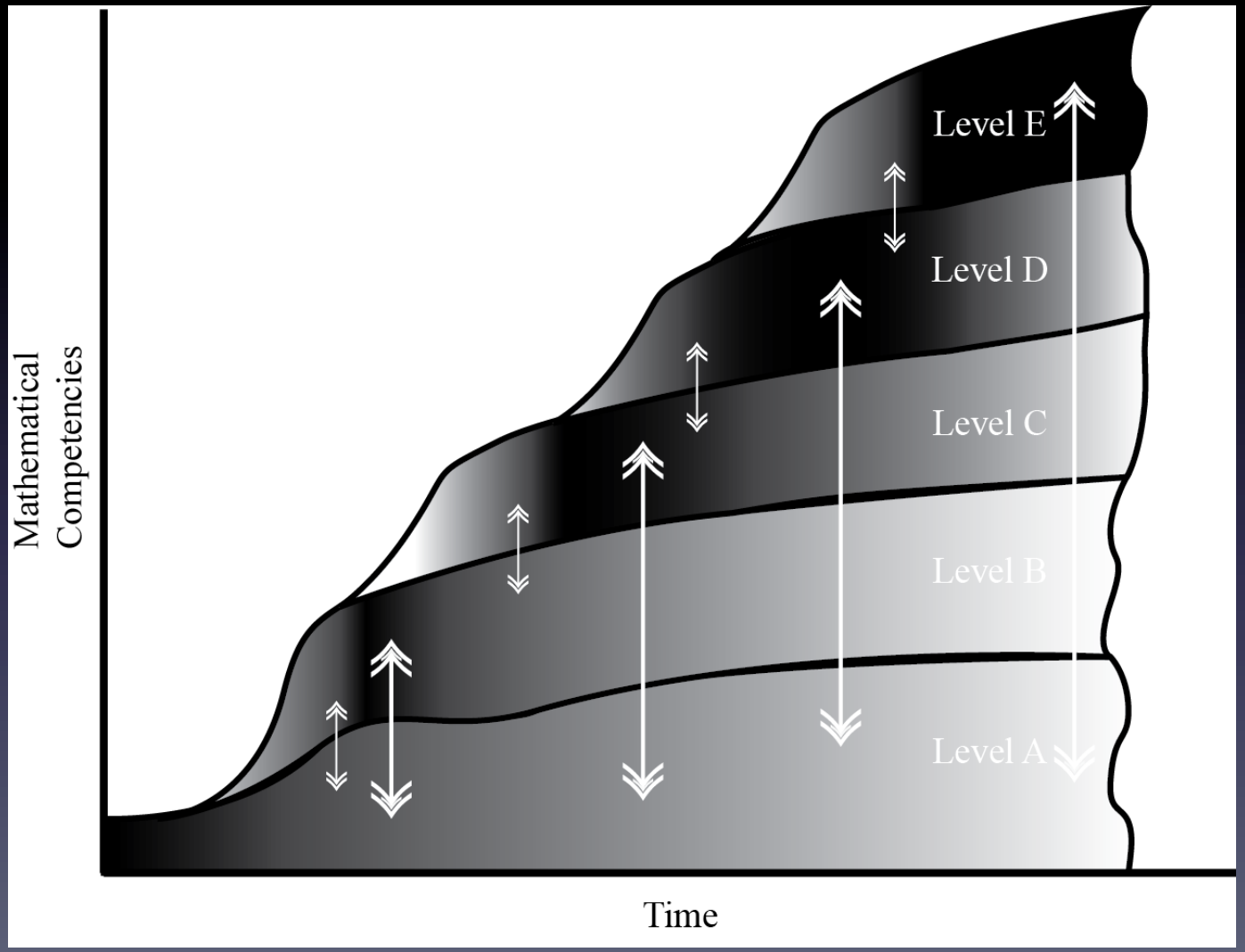
Increasing Sophistication

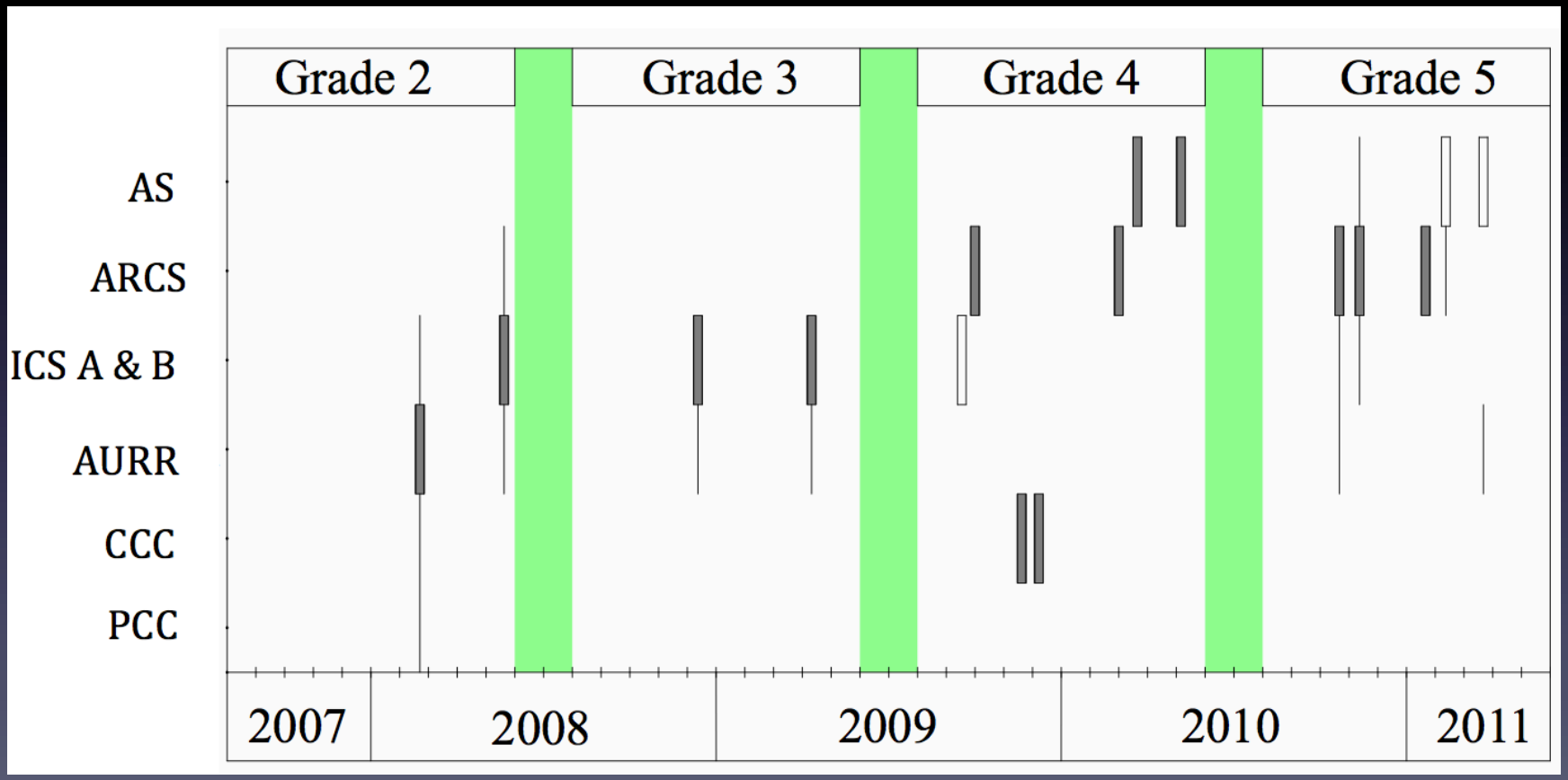
Longitudinal Case Study: Anselm

- The Midwest research site followed a cohort of 8 students from Grade 2 to 5.
- Today we focus on the case of Anselm's growth along the LT, showing each subsequent level of thinking as a coincidence with the schedule of our teaching episodes through four years:
  - We observed dominant thinking for this child at each level from AURR, through ICS, ARCS and on into AS
  - The relative duration of subsequent levels was balanced across this four-year study for this child.

Hierarchic Development: Theory of Levels



Anselm's Growth Chart
LT levels (revised Spring 2014)



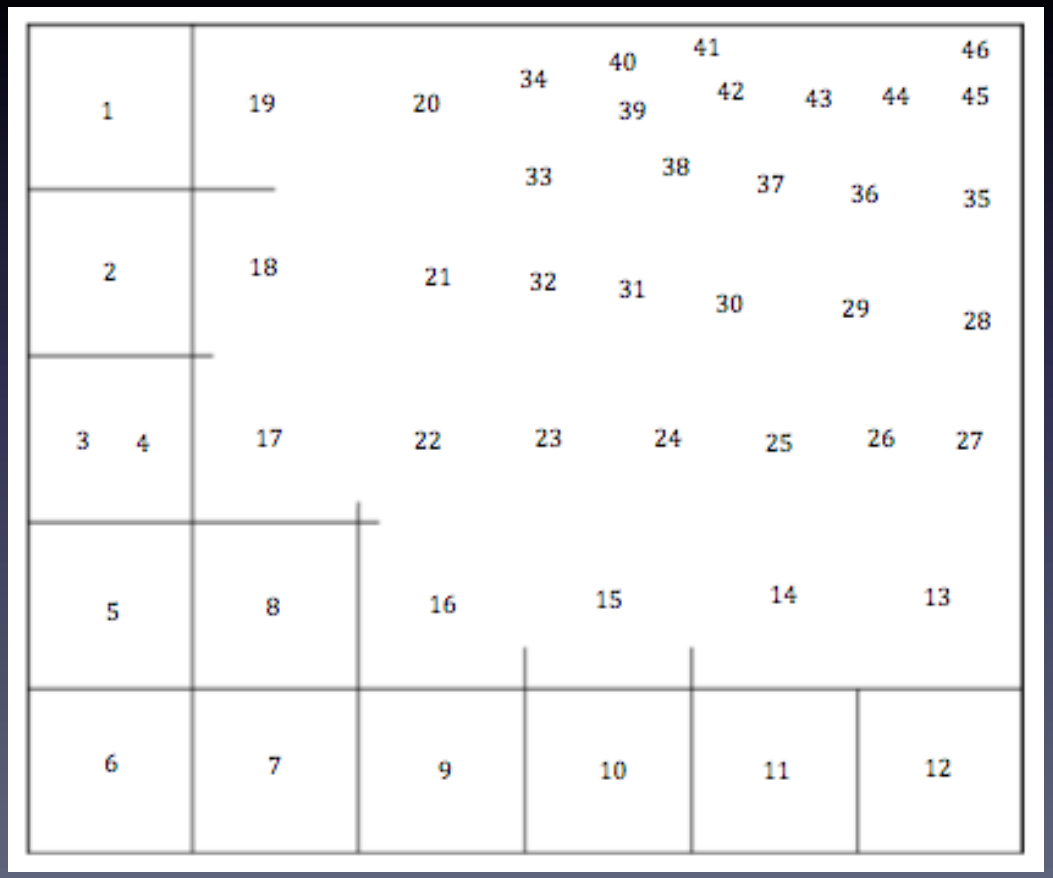
(Barrett et al, accepted, under review)

Complete Coverer and Counter (CCC)
Anselm: February of Grade 2

How many small squares would fit in this large square?

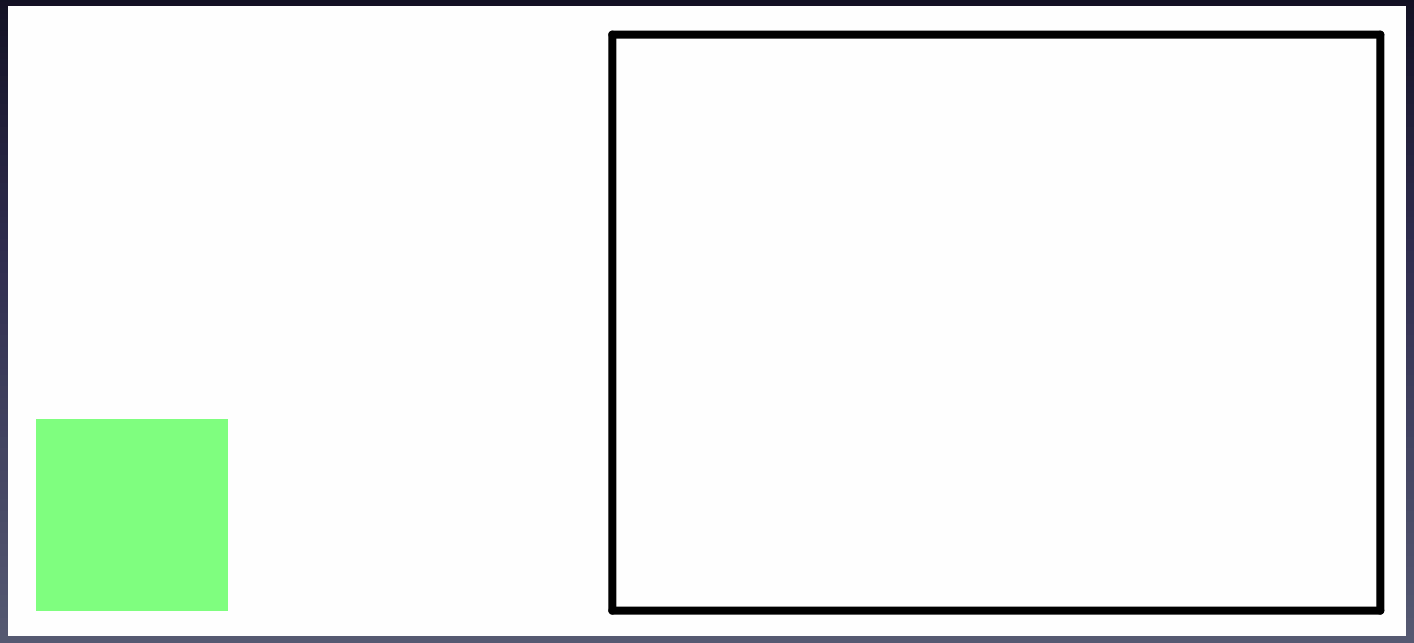
Anselm pointed and counted aloud from 1 to 46 as marked:

He was unsystematic in his counting of individual shapes, yet he demonstrated an explicit understanding that the entire region needed to be covered.



Area Unit Relater and Repeater (AURR)
Anselm: February of Grade 2

Draw what the rectangle would look like when completely covered with squares without moving or touching the square tile (green here).



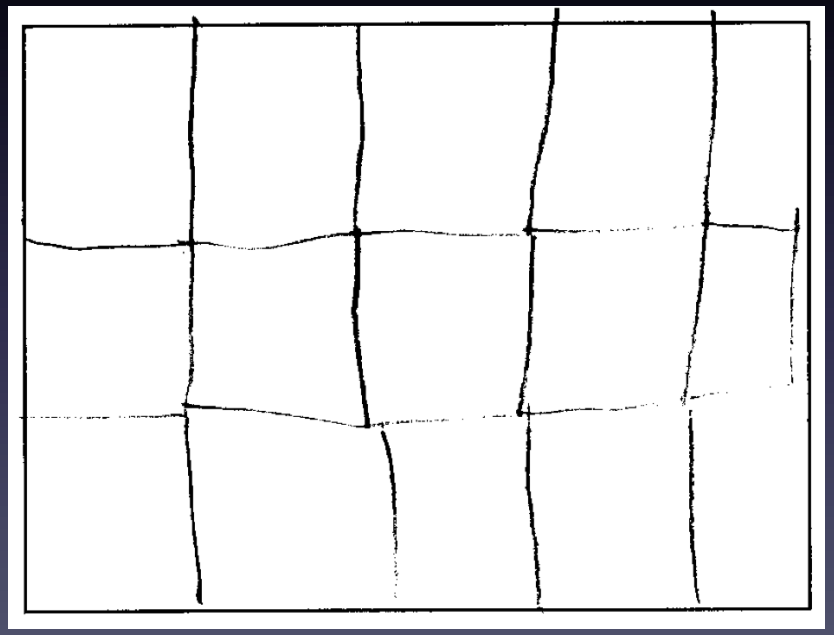
1 square inch

3 inches by 4 inches

Area Unit Relater and Repeater (AURR)
Anselm: February of Grade 2

Draw what the rectangle would look like when completely covered with squares without moving or touching the square tile (green here).

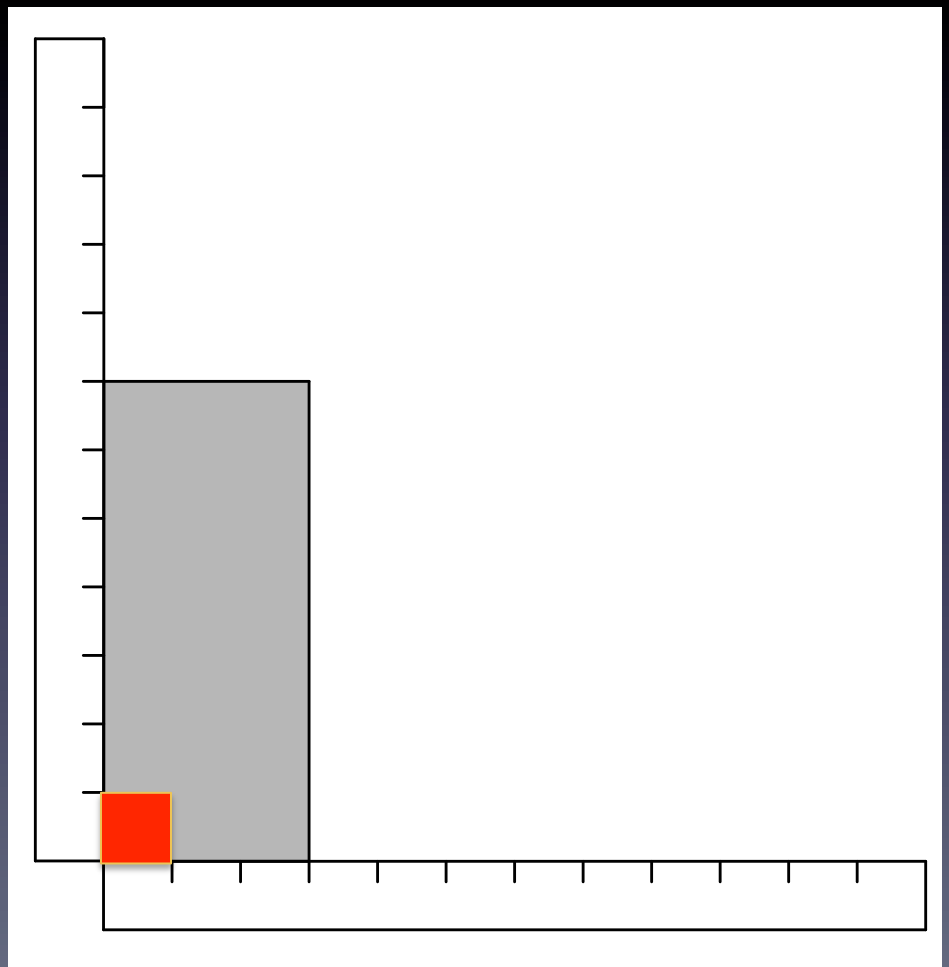
He drew a complete covering of individual units, one at a time, until the rectangular region was completely covered without gaps or overlaps. He ended up with a 3 by 5 array of approximately equal-sized units.



His drawing was guided only by the previously drawn adjacent units, and his alignment of the successive units was somewhat purposeful; he may have used the intuitive structure of a row. His drawing shows iteration of units.

Initial Composite Structurer A (ICS A)
Anselm: December of Grade 3

Interviewer: How many of these tiles would you need to cover the whole thing? ***Try without touching the tile***.

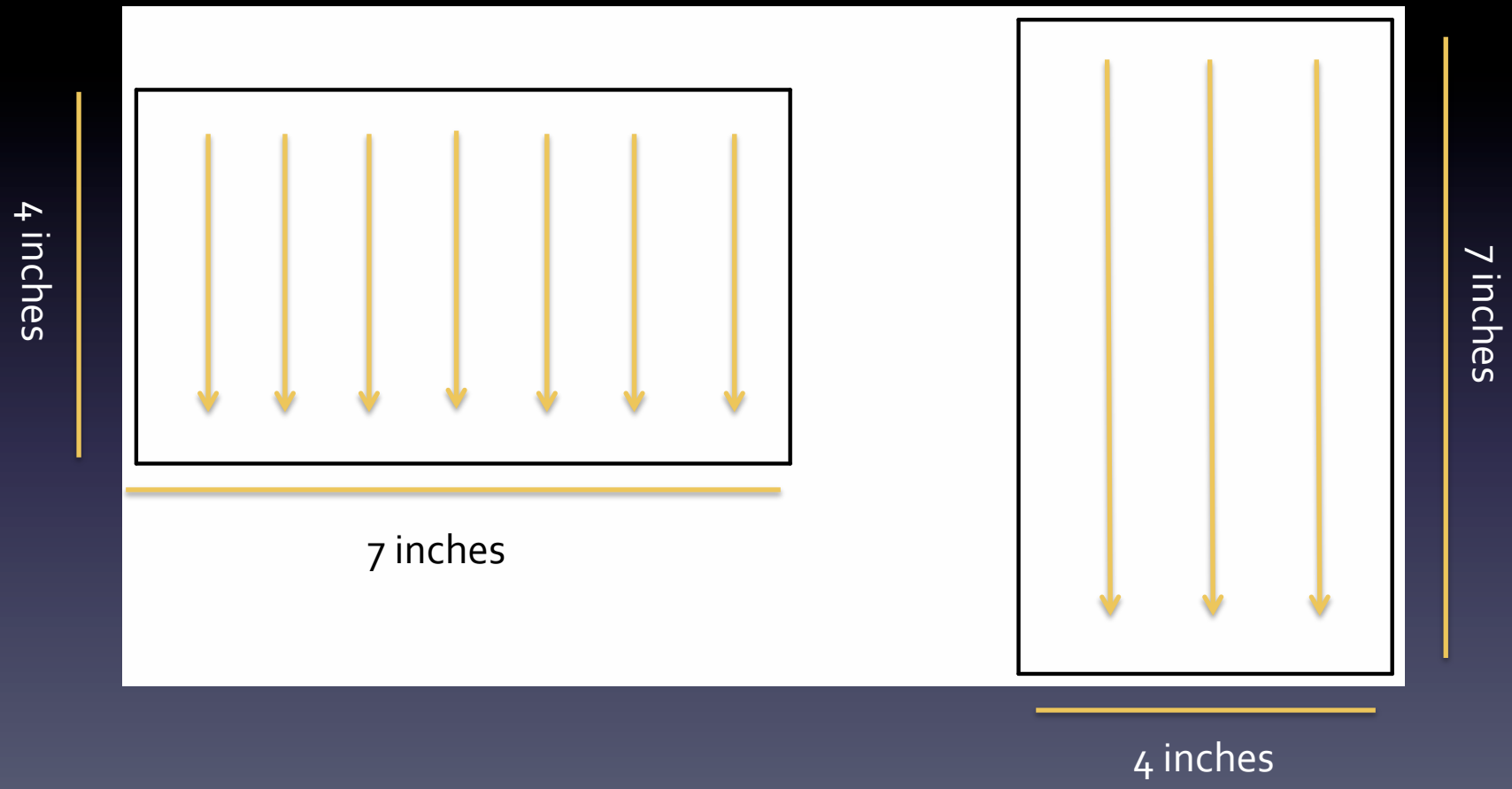


Anselm: All 3 of these [swept fingers across 3 rows] which would equal 9…
and then plus…
1 more 3 would be 12,
plus another 3 would be 15,
15, and then 16, 17, 18, [pause]
then 19, 20, 21;
21 squares would fit

He recognized a row as a unit of units but did not clearly indicate sets of these rows (counting 1 by 1 in part).

Initial Composite Structurer B (ICS B)
Anselm: April of Grade 3

Using a ruler to determine the area of a 4 inch by 7 inch rectangle

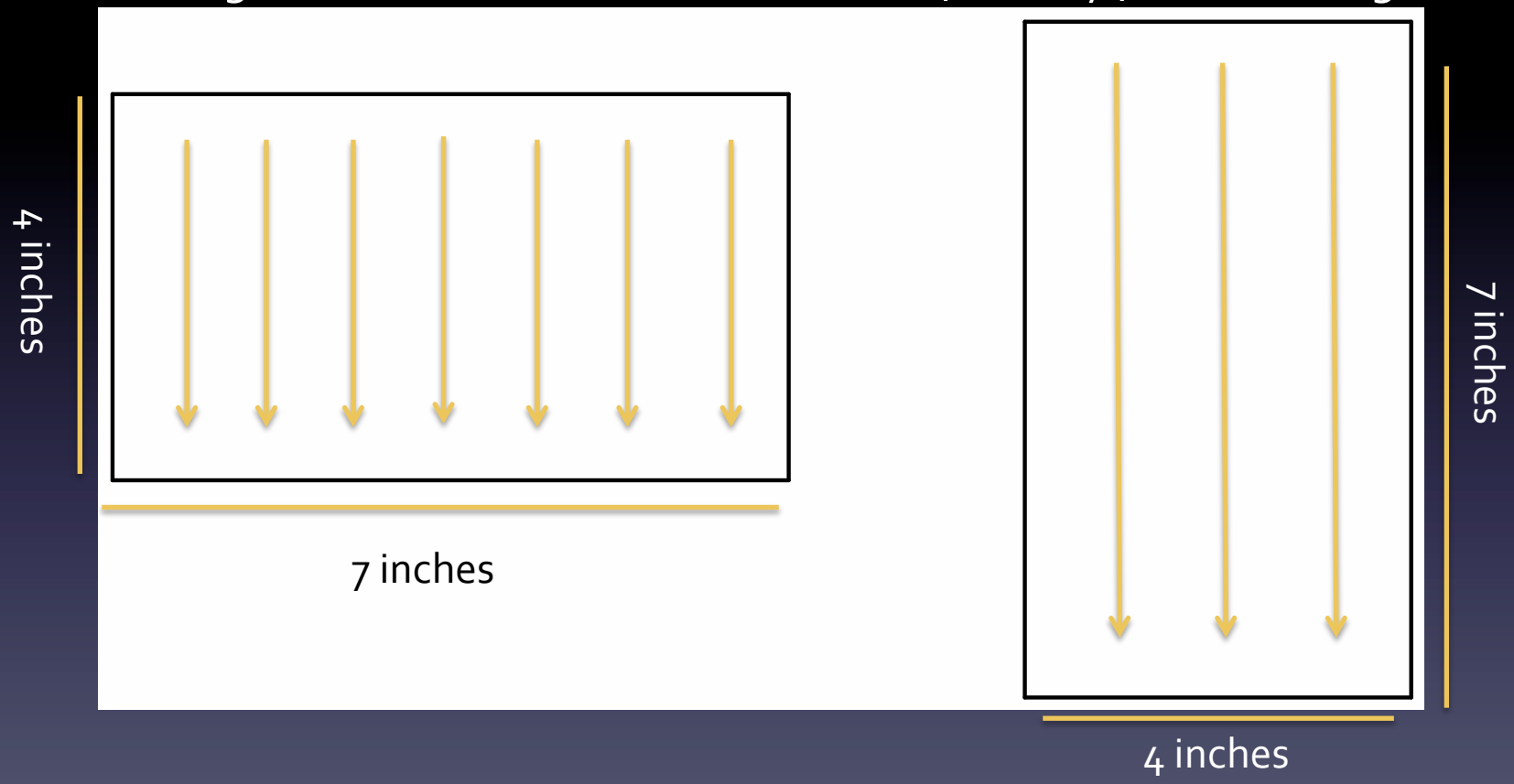


Counted 7 sets of 4

Counted 3 sets of 7

Initial Composite Structurer B (ICS B)
Anselm: April of Grade 3

Using a ruler to determine the area of a 4 inch by 7 inch rectangle



Here Anselm operated on a unit of units and indicated that he knew the dimensions of the rectangle were 4 and 7; however, he did not use them to determine how many area units would fit along a side in both dimensions.

Initial Composite Structurer: Two Sublevels?

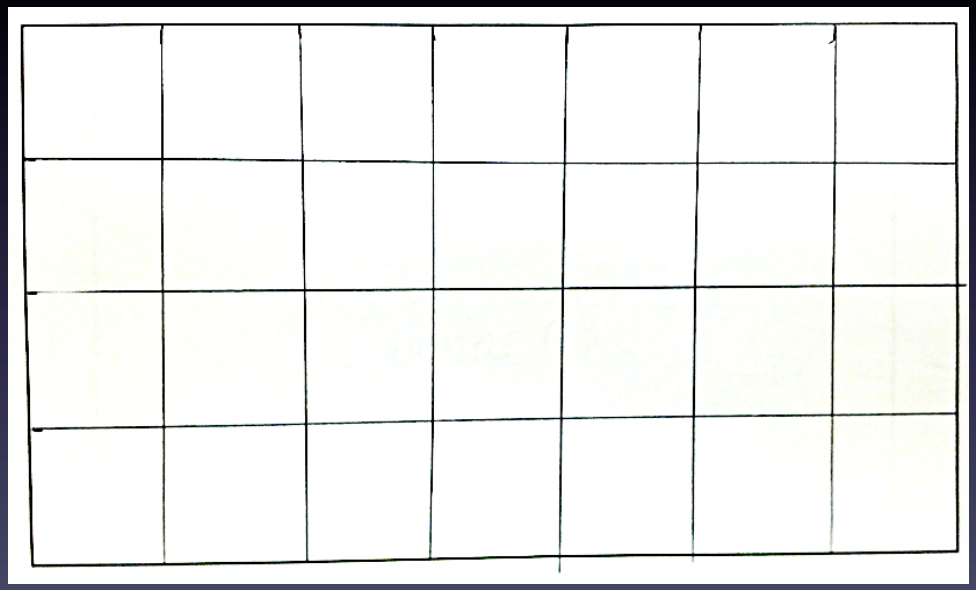
- Organizes counting, drawing, or moving objects in composite units (unit of units), yet without a consistent application of that structure to fit the linear dimensions.
  - *A. Early on, may be unaware of the congruence of rows, yet uses a unit of units (not necessarily a column, or a row) to determine a measure of a rectangular region. Structuring is very limited in scope; uses a mixture of individual units and units of units.*
  - *B. Later, structures a rectangular region as a set of rows (more comprehensively than before), attending to the collinearity of rows (expects rows to have the same number of units). Does not yet coordinate rows and columns of units with the linear extent of <u>both</u> length and width along the measured object.*

Area Row and Column Structurer (ARCS)
Anselm: Fall of Grade 5

Using a ruler to determine the area of a 4 inch by 7 inch rectangle

Anselm measured the length and width of the rectangle as 7 inches and 4 inches and used multiplication to determine the area.

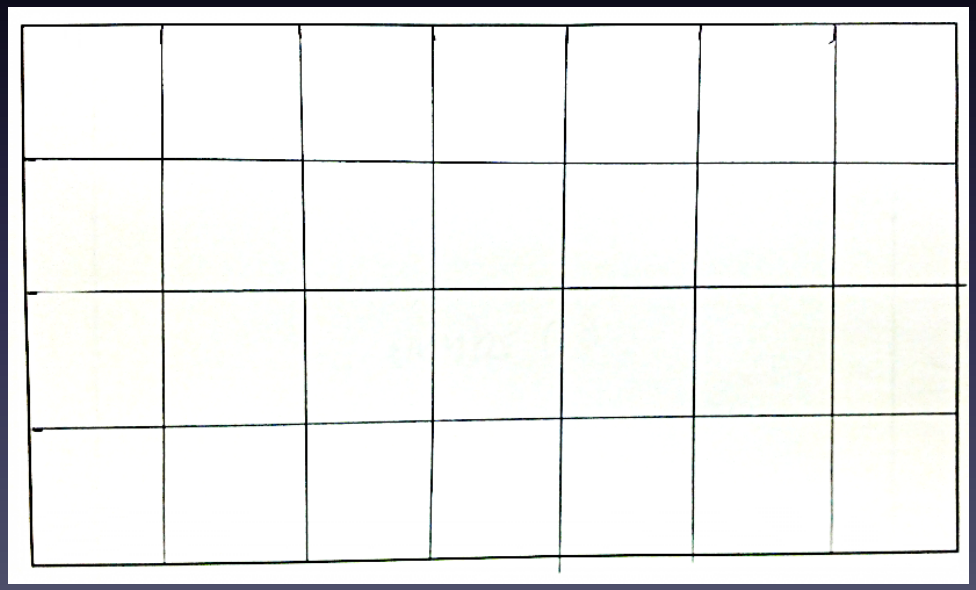
He said, "28 square inches will fit inside of this rectangle."



Area Row and Column Structurer (ARCS)
Anselm: Fall of Grade 5

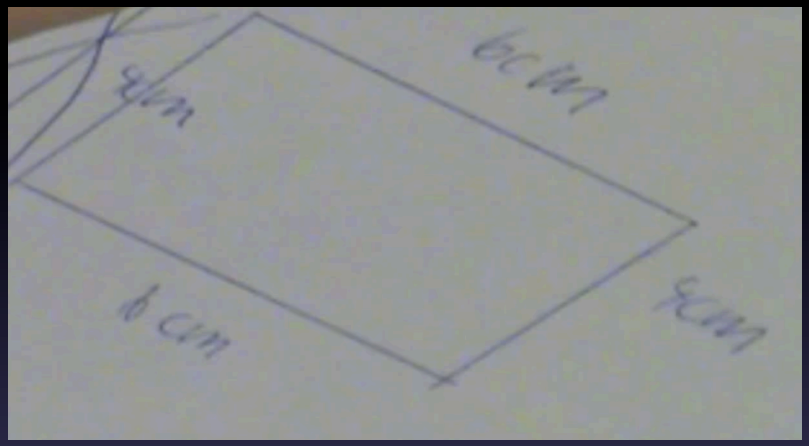
Using a ruler to determine the area of a 4 inch by 7 inch rectangle

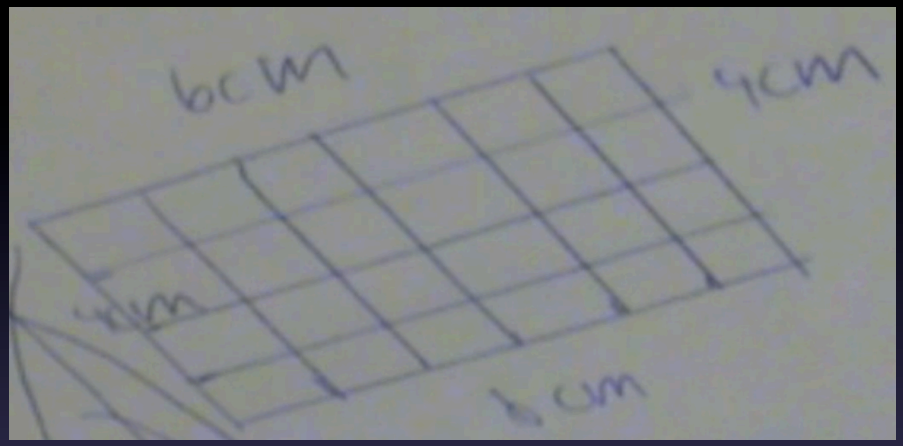
When asked to show how the 28 square inches fit, he subdivided the rectangle with parallel row and line column segments and applied the concept that the side length determines the number of area units that fit along that length in both dimensions.



Array Structurer (AS)
Anselm: March of Grade 5

Draw a rectangle that has an area of 24 square centimeters.





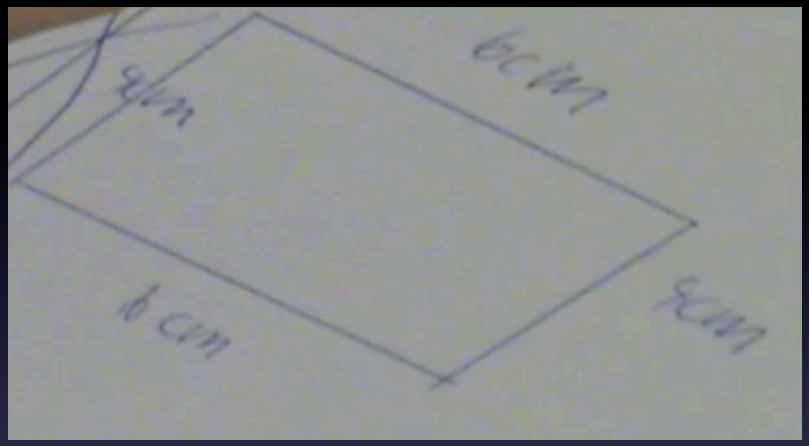
Anselm correctly drew a 4 cm x 6 cm rectangle.

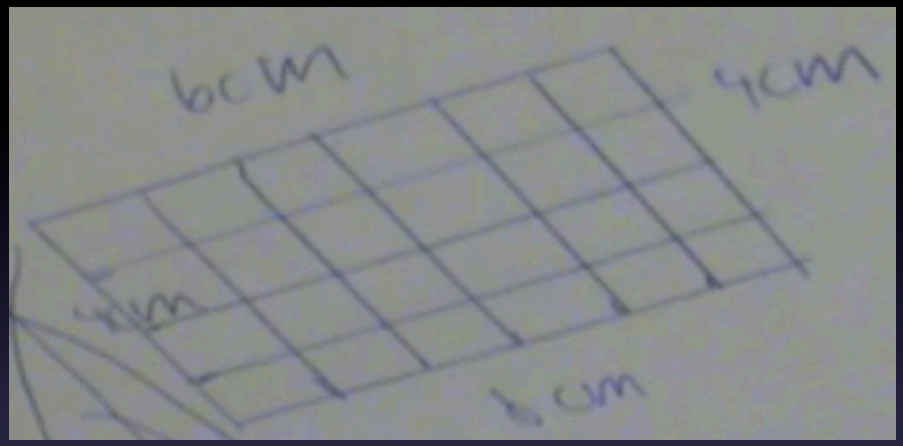
He was asked to determine the area of the rectangle that he drew. Anselm correctly answered, "24 square cm."
When asked to explain his thinking, he said that he multiplied 4 times 6 to get 24.

Next, Anselm drew the array, using the numbered tick marks on the ruler to constrain the placement of parallel row and column line segments.

Array Structurer (AS)
Anselm: March of Grade 5

Draw a rectangle that has an area of 24 square centimeters.





We take this as evidence of having a mental image of a spatial array and an understanding that the length of a line segment gives the specified number of area units that would fit along that segment.

Rationale for Changes to the LT

- We made shifts in grain size (ICS A and ICS B expands PRS level; collapsed PAQR and ASC into AQR level) to address central concerns of classroom teachers for identifiable accounts of progressions
- We renamed Side-to-Side Area Measurer to address evidence of a wider range of covering actions besides side-to-side actions (Now Physical Coverer and Counter)
- We found a broad range of children's thinking that fit the Area Conserver level. We have eliminated this category of thinking as one of the levels instead we take it as a *dimension* of thinking to help characterize several adjacent levels of thinking.

Revising the LT for Area Measurement: Collapsing Two Levels

Original LT for Area Measurement

- Pre-Area Quantity Recognizer
- Area Simple Comparer
- Side-to-side Area Measurer
- Primitive Coverer
- Area Unit Relater and Repeater
- Partial Row Structurer
- Area Row and Column Structurer
- Area Conserver
- Array Structurer

Revised LT for Area Measurement

- Area Quantity Recognizer
- Physical Coverer and Counter
- Complete Coverer and Counter
- Area Unit Relater and Repeater
- Initial Composite Structurer
  - A
  - B
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- Array Structurer

Revising the LT for Area Measurement: Recasting Levels

Original LT for Area Measurement

- Pre-Area Quantity Recognizer
- Area Simple Comparer
- Side-to-side Area Measurer
- Primitive Coverer
- Area Unit Relater and Repeater
- Partial Row Structurer
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Revised LT for Area Measurement

- Area Quantity Recognizer
- Physical Coverer and Counter
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Revising the LT for Area Measurement: Expanding a Level

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- Area Simple Comparer
- Side-to-side Area Measurer
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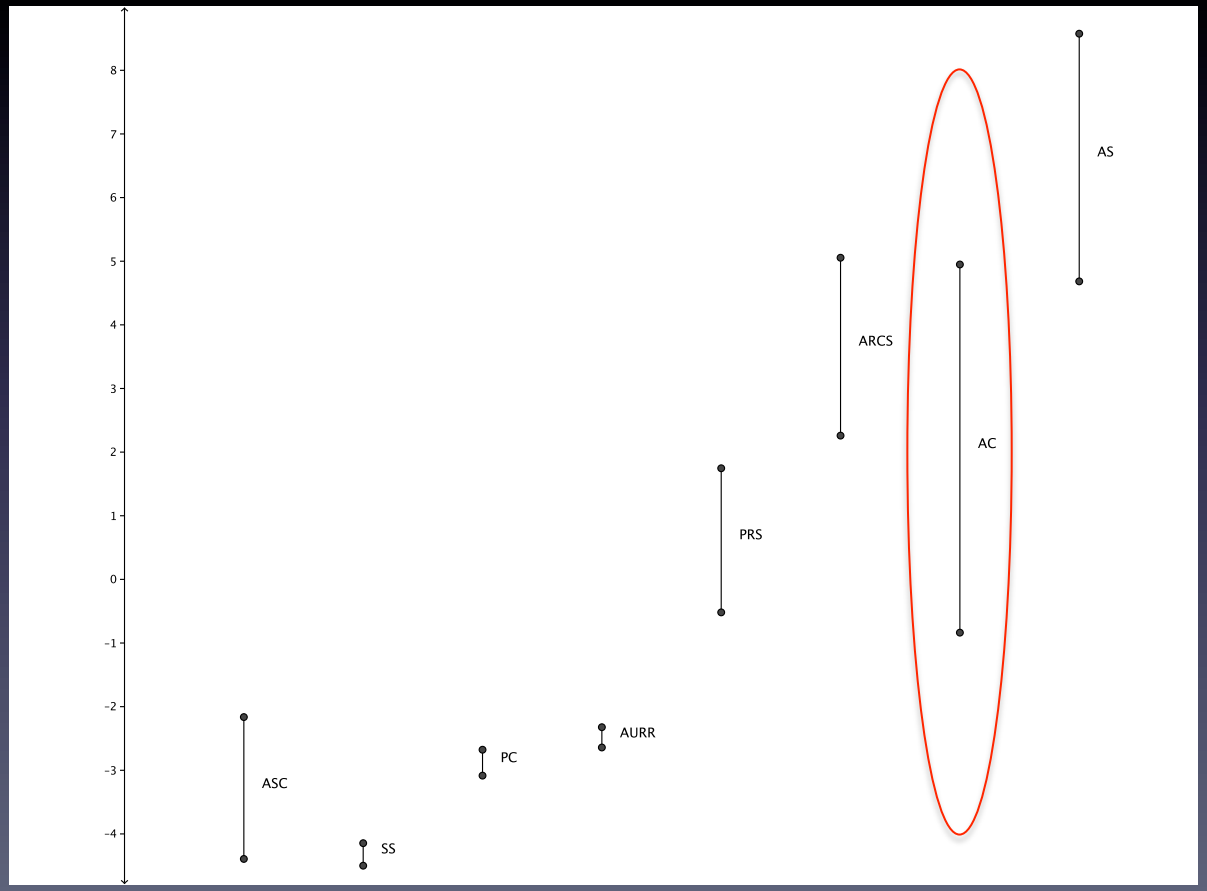
Revising the LT for Area Measurement: Deleting a Level

The Rasch analysis indicates that the responses of students at varying levels of expertise for area (vertical scale is a logit scale) still achieved similar results on the tasks we identified as area conserver tasks.

Given the dispersed performance, we eliminated this level from the LT. Now we treat it as a category across levels.



Revising the HLT for Area Measurement: Identifying Critical Dimensions of Area Measurement Activity

We noted a wide range of intuitive motivations for measures of area. We also noted a logical consistency with several aspects of volume measurement actions.

Here we list several critical aspects of area measurement activity that help us organize our accounts of student's misconceptions and struggles in the LT (column 3).

- Quantifying
- Drawing
- Comparing
- Producing
- Tiling
- Iterating
- Conserving

For example: Several Aspects of Thinking Exhibited *within* the Complete Coverer and Counter level

Complete Coverer and Counter (CCC) ***Drawing***. **Draws a complete covering of a specific region without gaps or overlaps and in approximations of rows.** ***Producing***. **When provided with more than the total number of physical tiles needed, can build a region of specified area (e.g., build a rectangle with an area of 12 from a pile of 20 tiles).**	**Applies explicit understanding that entire region must be covered with approximately rectangular shapes. Implicit visual patterning of multiple concatenations of rectangles. Guides placement of anchored units in rows, approximately aligned with parallel rows. The counting-all action schemes are used to enumerate the covering of a region.** ***Note on psychological foundation: Not yet able to understand the basis for the multiplicative transformation of lengths into area measurements.***	***Quantifying***. **In counting to a total, may count around the border and then systematically on the inside (possibly in a spiral, in an S pattern, or in approximations of rows). Initially, may not accurately count shapes, losing track or double counting. Later, may not lose track, counting more accurately, but without row or column organization.** ***Drawing***. **When drawing approximations of rows, often has errors in alignment of the shapes and does not yet recognize the need for equal-sized units.** ***Tiling***. **When asked to cover a region with physical tiles, may cover the whole region but use tiles of varying dimensions.** ***Iterating***. **When asked to cover without a complete set of physical tiles, may attempt to iterate but unable to maintain equal iterations.**

Rationale for Changes to the LT

- Shifts in grain size (ICS a and ICS b expands PRS level; collapsed PAQR and ASC into AQR level) to address central concerns of classroom teachers
- We renamed Side-to-Side Area Measurer to address broader evidence of covering actions (Now Physical Coverer and Counter)
- We found a broad range of children's thinking that fit the Area Conserver level, so we have eliminated this category of thinking from the levels and placed it as an aspect within a broad range of levels; we believe area conservation is a *dimension* of thinking rather than a level of thinking.

Thank you!

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