Children's Development of a Volume Calculation Algorithm for a Rectangular Prism with a Dynamic Virtual Manipulative

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Children's construction of a volume calculation algorithm for a rectangular prism with a dynamic virtual manipulative

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Measurement is an important component of elementary school mathematics because it links number and space and has practical applications in life. However, geometric measurement presents significant challenges for students (Battista & Clements, 1996; Barrett, Clements, & Sarama, 2017; Ben-Haim, Lappan, & Houang, 1985; Curry & Outthred, 2005). Although volume measurement is frequently introduced early in primary school, often prior to area, when students compare the amount of liquid filling different size containers (Smith & Barrett, 2017), it presents unique difficulties for students.

Unlike length or area measurement, students must coordinate the measurements from three dimensions to measure volume. Spatial thinking must also be integrated with enumeration strategies as children measure volume (Battista & Clements, 1996). Battista and Clements (1996) argued that students have a hard time coordinating the different arrays from volume and creating those arrays into a mental model. They even struggle when they are given the actual shape and real cubes to find an accurate measure. Curry and Outhred (2005) found students have difficulty finding volume when packing a cuboid with smaller cubes because of the issues with iterating the smaller cubes. When they attempt to do this, they often have overlaps or gaps within the cuboid. Ben-Haim, Lappan, and Houang (1985) found that students may ignore the 3D nature of the object and just attempt to count the visible faces additively, typically double counting edge cubes. Thus, it is important to characterize the development of enumeration strategies within the context of volume measurement using a virtual manipulative. In this research study, we sought to help students develop volume calculation algorithms by providing a constrained structuring of space through a dynamic representational intermediary.

1 Review of the Literature

Students begin to think about volume in four different ways: packing, filling, comparing, or building (Van Dine, et al., 2016?). Volume conceptualized as packing is putting cubes into a box to fill. It is a difficult idea for students as they focus more on cube faces, ignoring the 3D nature of the object. Volume conceptualized as filling considers a fluid unit of volume as filling up space. With this idea, students tend to focus more on the height than the area of the base. Volume conceptualized as building is typically developed with children's first experiences with volume when building and playing with things like blocks. They might say they used many blocks or a few blocks. Next, when students are given a 3-D object built out of cubes they can recognize and count the cubes on the faces. Volume conceptualized as comparing is when a child compares one 3D object with another and mentally aligns the two objects. Eventually, students should be able to coordinate flexibly between filling, packing, and building and additive and multiplicative comparison methods.

1.1 Unit Coordination

Students can engage in different strategies to find volume measurement. For example, they can use a formula such as length x width x height, they can count individual cubes, or they can use spatial structuring such as finding a layer and iterating it through the height. The shift for students from reason with single units to reasoning with units of units and units of units of units is an essential development (Smith & Barrett, 2017). In comparison between 2D and 3D structuring of space, Curry et al. (2006) found that students' competence to structure 3D lags behind their competence for to 2D structuring. When children first find the volume of a shape, they tend to focus on the surface area, ignoring the hidden cubes (Battista & Clements, 1996; Piaget et al. 1960). Students also tend to double count corners and edges (Battista & Clements, 2009).

Lehrer, Strom, and Confrey (2002) and Panorkou and Pratt (2019) have discussed students' success when they have students think about volume as the area of the base and then pulling the area to the given height of a cylinder or rectangular prism. Panorkou and Pratt (2016) found that when students used a virtual manipulative to explore the two- and three- dimensional shapes they did not have some of the difficulties that are commonly seen in measuring 3D shapes. The environment used was similar to how Lehrer, Slovin, and Dougherty (2014) had students think about area and sweeping area through a length. Our approach may combine some of the benefits of a virtual manipulative realized in these works, although we have maintained a cubic unit-based approach to volume calculation. We intend to examine the specific importance of relating the accumulation of unit objects, taken as a collection of discrete valued objects (units) to establish the entire prism as a quantity that is reported with a measured value. We expect to report on ways to support integrated representations of this accumulation and building process to learn about volume for right prisms.

1.2 Coordination of Two- and Three-dimensional Representations

Another difficulty with volume measurement recently found by Widder, Berman, and Koichu (2019) is, "Intrinsic visual features embedded in 2-D sketches depicting 3-D geometrical situations constitute visual stimuli that can evoke various subjective perceptions in different individuals" (pp. 520–521). Furthermore, Bako (2003) claimed that when students have to visualize an item 3-Dimensionally based on a 2-D sketch this causes a struggle with the teaching and learning of volume measurement. Many researchers corroborate the struggles that students have with using 2-D sketches and the difficulty it causes when trying to visualize the 3-D item (Christou, Pittaliz, Mousoulides, & Jones, 2005; Gutierrez, 1996; Kali & Orion, 1996; Widder & Gorsky, 2012). For example, when working with 2-D paper sketches, students will try to draw a cube (or unit) on a picture but instead draw a square or draw a net (Kara et al., 2012; Lehrer, Jenkins, & Osana, 1998). Widder et al. (2019) claim this topic of visualizing 2-D sketches of 3-D items needs more research. Because of the concerns with visualization, we wanted students to explore volume in a structured virtual environment to act as an intermediary to the 3-D setting.

1.3 Enumeration connected with space

It has been documented in past research that students finding prism volumes can count each individual unit, find the number of units in a layer and multiply with the number of layers or skip count, or use the formula (Battista & Clements, 1996; Cullen et al., 2017; Curry, Mitchelmore, & Outhred, 2006). Students typically begin by counting the units, then can move to counting the layers and multiplying the area or volume of the base by the number of layers (Smith & Barrett, 2017). This method can take time and students are often given the formula instead. It appears more efficient to give the formula but that precedes the students understanding of volume. Lehrer (2003) has found that students as young as grade 2 and 3 can make sense of finding the volume of a cylinder through thinking about the area of the base times the height but using very thin noncubic units.

1.4 Goals and Research Question

With this study, we sought to extend the research on volume measurement related to right rectangular prisms by exploring students' interactions with both 2-D sketches of right rectangular prisms and a dynamic virtual manipulative (DVM) environment with a scaffolded unit structuring scheme. We designed a dynamic measurement approach and created that program to document children's growth in developing a volume measuring algorithm for right rectangular prisms.

We believed through repeated opportunities to engage with a virtual manipulative and through providing them a constrained structuring of space, children would be able to develop their ability to develop volume calculation algorithms. Our goal was to document how the students developed their understanding of volume calculation to the point of applying a self-developed algorithm. With these ideas in mind, the following research question guided our work:

Research Question: How do students develop a volume calculation algorithm for rectangular prisms using a constrained unit structuring scheme in a dynamic virtual manipulative environment?

2 Theoretical Framework

To help answer our research question, we first look to hierarchic interactionalism and follow by examining learning trajectories (LT). We looked to two different research groups' works (Battista, 2010; Barrett et al., 2017). With Battista's (2010) work, we focus on what is a LT, what is the purpose, and how our research can be guided by Battista's idea of LTs. Next, we use Barrett et al. (2017) as a guide to the specific levels of volume thinking the students displayed and as an analysis tool to document the change the student displayed.

Hierarchic interactionalism (HI) "indicates the influence and interaction of global and local (domain specific) cognitive levels and the interactions of innate competencies, internal resources, and experience" (Clements & Sarama, 2008, p. 464). This means students first make sense of ideas intuitively, then through language and metacognition, and last the child shows signs of understanding. According to HI students develop their level of thinking through a developmental progress. They have a dominant level but can reach into more than one level with more or less sophisticated reasoning. The learning progression has been defined by Battista (2010).

A learning progression (LP) consists of levels of student reasoning based on sophistication (Battista, 2010). Battista (2010) defined a LT:

> as a detailed description of the sequence of thoughts, ways of reasoning, and strategies that a student employs while involved in learning the topic, including specification of how the student deals with all instructional tasks and social interactions during this sequence. (p. 61)

Battista argued that a trajectory must give a description of the instruction and have a fixed sequence of different learning tasks. He claimed the purpose is to help teachers "understand, plan, and react instructionally, on a moment-to-moment basis, to students' developing reasoning" (p.).

A level in a LT is a where a certain type of cognition occurs over a period of time (Battista, 2010). These levels are in order of sophistication in the student's reasoning. When the students are at a level, their cognitive structures and processing are capable to make sense of and think about the given topic. The time period that they are at the level can be different and they can spend significant time in shifting to the next level. When a student is able to give an answers for the level they are on but sometimes reach into the next level with their answers, they are at the time of transition between the different levels.

In Battista's (2010) work, the students develop through levels beginning with the pre-instructional reasoning that students typically possess. The levels end with formal mathematical concepts.

2.1 Hierarchic interactionalism

To guide our thinking about LT, we also focused on Hierarchic interactionalism. We have established hypothetical learning trajectories (HLT) to characterize such development and to support the development of curriculum and enhance teacher knowledge (Barrett et al., 2017). Similar to Battista's (2010) work, a trajectory includes the mathematical learning goal, the thinking and learning in which students might engage, and the pertinent learning activities to support growth from one level to the next (Simon, 1995). We employed a trajectory for volume measurement to classify student growth patterns and to design and test a learning activity. In this study, we designed and tested a DVM intended to support organizational structuring of volume units.

The HLT for volume measurement (Barrett et al., 2017) used in this study is revised from Sarama and Clements' (2009) LT for volume measurement. We used the LT for volume measurement from hierarchic interactionalist perspective on development and learning (Clements & Sarama, 2007). The HLT was extended through a 4-year longitudinal study of elementary students in a teaching experiment, as reported by Barrett, Clements, Sarama, Miller et al. (2017).

2.2 Levels of an LT for Volume Measurement

The earliest, least sophisticated level of the developmental progression (Volume Quantity Recognizer) for volume measurement begins with students recognizing that they can fill or pack space, use more blocks or fewer blocks, identify objects as big or small, and compare volumes as either bigger, smaller, taller, or shorter. Students at the second level (Volume Filler) can compare an object by two dimensions, can fill a larger container with smaller containers, can pack cubes into a box, and can recognize and count cubes but it may be on multiple faces. Students in the third level (Volume Quantifier) can quantify volume in ways such as the number of scoops to fill a space, packing completely with cubes, and compare using three dimensions. Students in the fourth (Volume Unit Relater and Repeater) level can relate the size and number of units explicitly, count cubes when finding the volume and not the faces, accurately pack and count the units to fill a space, and compare volume using different numbers of units. Students in the fifth(Initial Composite 3-D Structurer) level can understand the number of cubes or cubic units as a measure of capacity, visualize and operate on composite units such as rows or columns and can account for hidden cubes, recognize that a box is half full and visualize the remaining rows or columns, and develop more accurate counting strategies. Students in the sixth level (3-D Row and Column Structurer) can coordinate flexibly with filling, packing, and building, and operate fluidly and flexibly on units, units of units, and units of units of units. Students in the final level (3-D Array Structurer) have an abstract understanding of the rectangular prism volume formula.

For this study, we focused on the levels Volume Unit Relater and Repeater (VURR), Initial Composite 3-D Structurer (VICS), and 3-D Row and Column Structurer (VRCS) because we wanted to find how to promote growth through the initial composite level of thinking and help students begin using mental models of rows and columns, taken together to structure space as they would measure rectangular prisms. We predicted as students initially attempted to organize and structure volume units, they would do so without coordinating the set of cube faces on a prism. This would be VURR. Next, we thought they would be able to find a row using only one-unit cube, rows to fill a layer with only seeing one row or use layers to find the volume but not find the total volume. This would be level VICS. Lastly, we hoped that students would be able to integrate a set of locally coordinated structures within a global structure to find the volume of a prism. This would be VRCS. But this experiment was needed to expand our knowledge of effective instructional interventions for these given levels in the trajectory

3 Methods

In this study, we wanted to observe how students develop a volume calculation algorithm for rectangular prisms when beginning with basic or limited volume unit counting techniques through the development of composite unit structuring schemes. We used a microgenetic method to help us observe children through this process (Siegler & Svetina, 2006) because it offered the opportunity to closely examine the conditions for learning. By using a microgenetic methodology we could document the circumstances preceding a conceptual change, the change, and the following results. There are three conditions of a micgrogenetic method that allow insight on growth: (a) observations that span an entire period of rapidly changing competence, (b) a high density of observation within this period, relative to the rate of change; and (c) intensive analysis of observations of changing performance that illuminate the processes that give rise to change (Siegler & Sventina, 2006). In order to produce conditions conducive to rapid change, we developed an iterative treatment involving the coordination of a traditional volume problem presented two-dimensionally on paper and a dynamic virtual manipulative (DVM) that modeled composite unit structuring.

3.1 Dynamic Virtual Manipulative Design

We developed and used a dynamic virtual manipulative (https://www.geogebra.org/m/FgQVdDTb) to help students recognize that edge lengths can be used to predict the number of cubes along an edge and develop their use of composite units for structured counting of volume. We hypothesized that the manipulative would highlight the efficiency of enumerating composite units by constraining how students interact with increasingly complex units: first single cubes, then row collections of cubes, then layered collections of rows of cubes. By constraining the structuring scheme, we expected to produce rapid growth through modeling of an efficient scheme. Further, we expected to prompt for a correspondence between an edge length and the number of an appropriate unit along that edge.

The DVM, created in geogebra, included 12 similar volume problems. In each problem, the computer showed a rectangular prism with 6 visible edges labeled with a numerical value to indicate length (see Figure Q) and a single cube outside the prism with edge lengths of 1 unit. A question accompanying each problem read, “The volume of the small cube is one cubic unit, what is the volume of the larger solid?” Students cycled through problems using “Next Task” and “Previous Task” buttons. The primary action of the students was to use three sets of two buttons each (one showing an addition symbol and one showing a subtraction symbol), which would iterate units or composite units to build a rectangular prism. A green set of plus and minus buttons increased or decreased respectively the iteration of units or composite units by one along the traditional *x*-axis. A blue set of plus and minus buttons increased or decreased respectively the iteration of units or composite units by one along the traditional *y*-axis. A red set of plus and minus buttons increased or decreased respectively the iteration of units or composite units by one along the traditional *z*-axis.

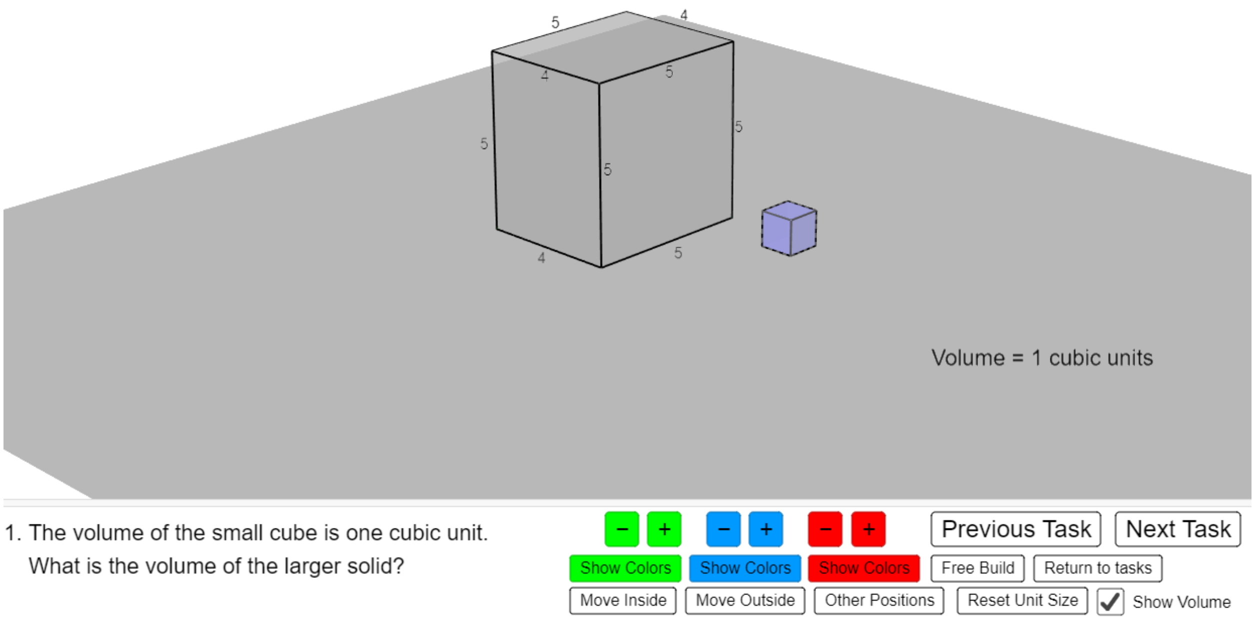


Figure Q. *Dynamic Virtual Manipulative Environment*

We asked students to use these sets of buttons in succession first green, then blue, then red without returning to previous colors. When the student used the green set of buttons one cube at a time would be added or subtracted from the set of cubes, creating an *n* x 1 x 1 row of cubes (see Figure P). Then, when the student used the blue set of buttons new *n* x 1 x 1 rows would be added (or subtracted), one row at a time, resulting in an *n* x *m* x 1 layer. Then, when the student used the third set of buttons new *n* x *m* x 1 layers would be added (or subtracted), one layer at a time, resulting in an *n* x *m* x *l* prism. Using the buttons in a different order would result in a similar process, but with each composite unit oriented differently. A label next to the prism (with a toggle on and off check box) indicated the volume of the built prism in unit cubes.

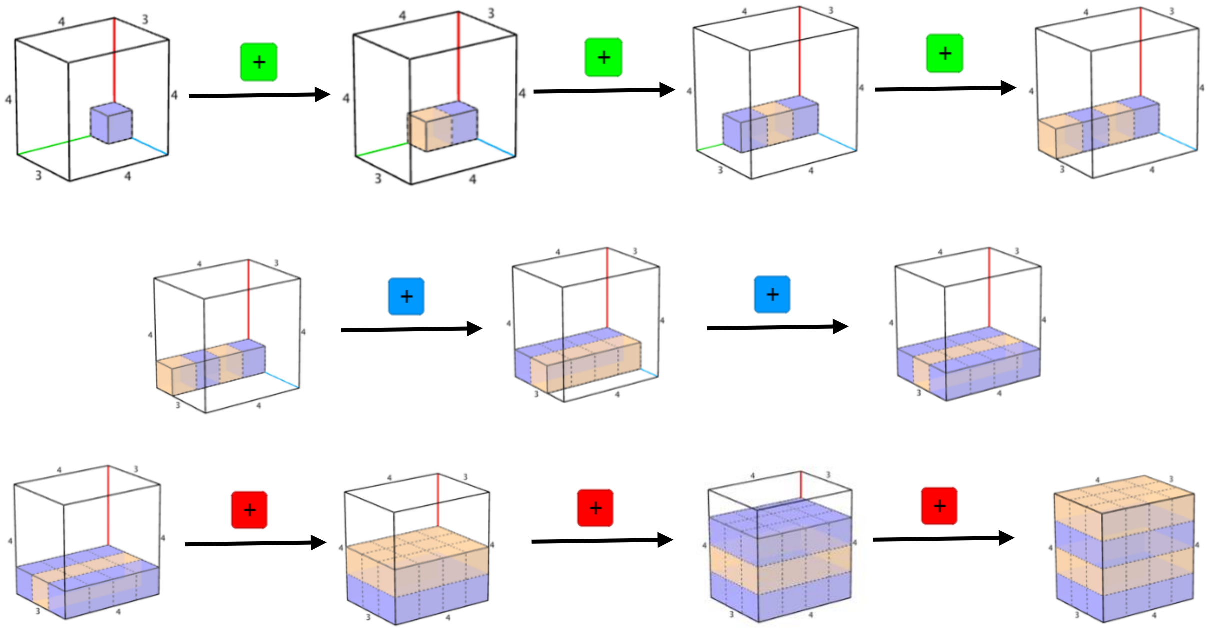


Figure P. *Actions of the DVM in Response to Usage of Green, Blue, Then Red Plus Buttons*

Move inside and move outside buttons moved the small cube inside the rectangular prism and back outside the prism respectively. Moving the cube inside the prism changed the representation from a solid representation to a wire frame representation with the small cube was positioned in the back, bottom corner of the prism and the back edges changed color to match the orientation of change instigated by the green, blue, and red buttons (see Figure P). That is, if we considered the back, bottom corner the origin, the *x*-axis was green, the *y*-axis blue, and the *z*-axis red. Additional buttons provided other functionality, but those buttons were used minimally in this study. The functionality of geogebra allowed students to zoom in and out, change the location of the prism on the screen, and reorient the prism.

3.2 Protocol

One or two members of the research team interviewed each student five times in February and March. Each interview lasted between 10 and 30 minutes. We conducted each of the five interviews on different days and interviewers rotated students. We ran a pilot test of the intervention with a similar population prior to beginning the study to refine the interview protocol and students' interaction with the DVM.

During the first session we administered a pre-assessment (not part of this analysis) and provided an orientation to the DVM. We asked students to predict what would happen when they clicked the green plus button, then had them click the button once. We then asked what would happen if they clicked the button again and allowed them to click the button a second time. We repeated this process several times with the plus button, then with the minus button, followed by the blue and red plus and minus buttons. Students also tried the "Move Inside" and "Move Outside" buttons and the spin feature that allowed them to rotate the prism about *the* z-axis.

In sessions two, three, and four, students completed three volume calculation tasks. In session five, students who consistently calculated volume correctly prior to session five completed two tasks and a transfer test, but students who had not consistently calculated volume correctly prior to session five completed three tasks and did not do the transfer test. The transfer test is not included in this analysis because not all students took the test.

For each of the eleven or twelve tasks, students completed a traditional paper volume problem (what we refer to as the paper portion) and the same problem using the DVM (what we refer to as the DVM portion). On the first task in each session, we showed a physical prism and a unit cube (wooden and plastic blocks) with the same dimensions as the prism in the paper portion and DVM portion. We told students that the representations on the paper and DVM were meant to represent these physical objects. We measured the dimensions of the physical prism in centimeters and told students that the numerical labels on the paper represented these measurements, asking students to confirm that the dimensions matched and the figures were representative of the physical prism. We refrained from including the centimeter label on either representation in an attempt to alleviate any confusion caused by a perceived difference in size of the units in each representation. On the second and third task in each session we did not show a physical prism or measurement but stated that we intended the paper and DVM to represent a physical prism.

For the paper portion, we provided the student with a representation like the one shown in Figure O and read the question printed on the paper, “The volume of the small cube is one cubic unit. What is the volume of the larger box? Please make a record of your work.” We did not provide any supports on this portion except that we offered to act as a calculator, if needed. Students recorded their result in a box at the bottom of the paper.

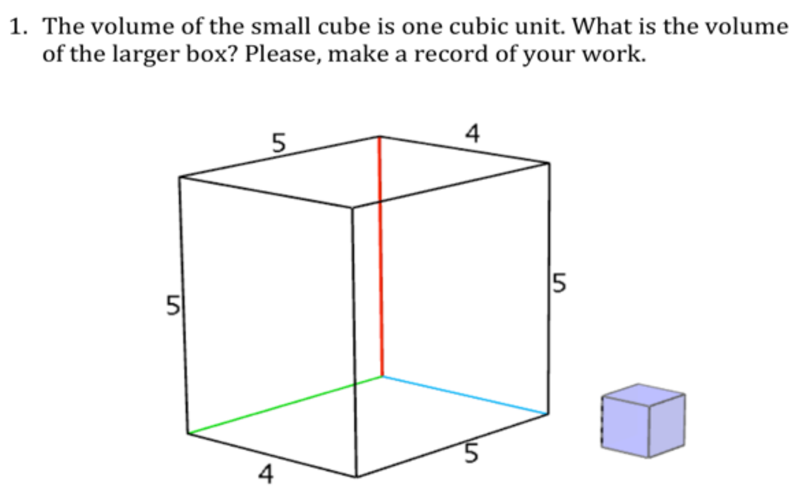


Figure O. *Traditional Volume Problem on the Paper Portion*

After the student provided a volume for the paper portion, we asked him or her to confirm that the virtual manipulative was showing the same prism as shown on the paper portion. The first thing we encouraged the student to do was to move the unit cube inside the frame of the rectangular prism.

Next, we told the student he or she would use the green buttons first, then the blue buttons, and then the red buttons to fill the prism. We explained that the buttons should be used in succession without returning to previous sets so that progressing through all three sets of buttons would fill the prism. We asked the student to predict saying, "How many cubes will there be when you are all finished with the green buttons?" If the student gave a correct prediction, we asked, "How many cubes will there be when you are all finished with the blue buttons?" Again, if the student gave a correct prediction, we asked, "How many cubes will there be when you are all finished with the red buttons?" If students gave an incorrect prediction for any set of buttons, we did not ask about the subsequent set. Instead, we told the student to use the green set of buttons to check his or her prediction. After producing a row with the green buttons, we asked the student to use the blue buttons only if he or she had already correctly predicted the number of cubes in a layer (produced using the blue buttons). But if the student had not made a correct prediction about a layer we asked, "How many cubes will there be when you are all finished with the blue buttons?" If this prediction was correct, we also asked, "How many cubes will there be when you are all finished with the red buttons?" After producing a layer, we followed the same procedure of asking for a prediction about the use of the red buttons only if a correct prediction had not already been provided. These restrictions created an environment conducive to students' growth in spatial structuring.

When students completed the use of the DVM, they recorded the volume as reported by the computer in a box on the bottom of the paper next to their paper result. Then, we asked, "What do you think the actual volume is?" and asked them to record that on the paper in a third box. Then, we moved on to the next task, completing three tasks in each interview session.

3.3 Participants

We applied this intervention with third (14 students) and fourth (17 students) grade students from a private school in the Midwest. We chose students in third and fourth grade because they were likely to show growth in volume structuring. Battista and Clements (1998) found that most grade five students were able to characterize a rectangular prism as a series of layers of rows and columns of cubes but only 20% of grade three students were able to do this. The students in the study had yet to been formally introduced to volume but the fourth-grade students had previously studied area.

3.4 Data collection

We video recorded all interview sessions, including pre and post assessments. A screencast also captured all the activity with the DVM along with a video of the student's face and audio. In addition, the interviewer recorded all student responses for volume calculations and predictions on a record sheet, and we retained all written work from the student.

3.5 Analysis

Our analysis occurred in five stages. First, we looked for overall growth in the population throughout the interview tasks. Second, we identified growth among individual students. Third, we identified approaches to the integration of paper and DVM portions of the tasks. Fourth, we confirmed that growth indicated level progressions in a volume learning trajectory by identifying students' demonstrated level of thinking on each portion of each task. Fifth, we closely investigate the growth that occurred across levels by using an emergent coding scheme focused on students' challenges, emerging understandings, and elements of feedback that led us to patterns of growth among the participants.

3.5.1 Overall Growth

We identified overall growth in the population based on increasing accuracy in predictions and volume calculations throughout the interview tasks. We counted correct volume calculations on the paper portion across all students on each task to determine if more students answered the question correctly as the task number increased. We also counted the number of correct predictions across all students when seeing only a small cube on the DVM portion of each task to determine if more students made correct predictions as the task number increased. See section 4.1 for growth results in the population.

3.5.2 Individual Growth

We used a similar process to identify growth in prediction and volume calculation among individual students. However, we counted predictions when seeing a cube, when seeing a row, and when seeing a layer. Using this empirical evidence of growth, we found that some students did not demonstrate growth empirically because their prior knowledge in volume was such that the primary purpose of the intervention was previously or immediately accomplished. We found that other students did not demonstrate their growth empirically within timeframe of the study, because the intervention was unsuccessful at producing empirically verifiable growth. However, we found many students demonstrated empirically verifiable growth throughout the study as measured by accurate volume calculations and prediction. The first group was excluded from further analysis because any growth that occurred was beyond the interest of this study. The second set of students were informally analyzed to probe for possible analytic avenues that might illuminate barriers to growth. Because this informal analysis did not yield productive pathways, this group was also excluded from further analysis. Thus, the remainder of our analysis focused on the third set of students because we were interested in these students' growth in volume calculation and they demonstrated growth in this area.

3.5.3 Approaches to Integration

Based on the analysis of individual growth, we noticed that students took different approaches to the integration of the paper and DVM portions of the tasks. We identified these different approaches and demonstrated them with graphical representations of predictive patterns. See section 4.2 for results relating to the integration of the two portions of each task.

3.5.4 Level Progressions

Because individual students demonstrated increasing accuracy in volume calculations and predictions, we hypothesized that these students progressed one or more levels on a learning trajectory for volume. To confirm this, two independent researchers coded students' demonstration of level-based thinking both for the paper portion and DVM portion throughout all tasks. Any discrepancies among the independent coding were discussed and reconciled by the two researchers. This analysis showed a progression in student development that mirrored the previous evidence of growth. However, this did not yet help illuminate the mechanisms of progression from level to level.

3.5.5 Patterns of Growth

To closely investigate the growth that occurred across levels we developed an emergent coding scheme. We focused the coding scheme around the challenges students faced, their emerging understandings, and the feedback that was instrumental in propelling students through a challenge to a new understanding. One researcher developed an initial set of codes by coding multiple sessions. Then, a second researcher used the definitions to code one of the coded sessions independently, adding new codes as necessary. The two researchers worked together to identify and rectify coding discrepancies by adding new codes to the scheme, condensing codes to eliminate redundancy, and redefining codes to clarify appropriate application by multiple coders. This process occurred iteratively across multiple sessions until the coding scheme reached a level of saturation at which no new codes showed up for multiple sessions. Then, the two researchers continued to utilize the scheme to code the remainder of the sessions independently and to check the coding of all sessions previously coded. All interview sessions were coded independently by both researchers and all coding discrepancies were discussed and reconciled by the researchers.

Based on these codings, the two researchers identified four themes, which we call fundamental threads of growth. Each of these threads were independently important for students' volume calculations and predictions, but coordination among the threads was necessary for complete understanding of volume. In addition, the growth in the thread progressed interdependently, but not in a single, predictable fashion. We labeled these four emergent threads: Structure, Interpretation, Representation, and Enumeration.

Structure refers to a student's structural thinking. One of the key components within this thread was a student's development of composite units. Throughout the interview sessions we observed students expand their capacity to compose unit cubes into a row, a row into a layer, and a layer into volume.

Interpretation refers to a student's interpretation of the question of volume. Key components within this thread include a student's interpretation of volume and their coordination of the questions asked in the paper and DVM portions of the tasks. Throughout the interview sessions we observed students reinterpret volume as a question of space filling that can be counted using unit cubes and coordinate their interpretation of the DVM and paper portions of each task as equivalent questions.

Representation refers to a student's conceptualization of the various representations presented in the problem context. Key components within this thread include a student's interpretation of the length labels, as well as his or her coordination of the physical model, the static two-dimensional drawing presented in the paper portion, and the DVM representation. Throughout the interview sessions we observed students recognize the length labels as a representations of how many volume units fit along an edge and coordinate the disparate representations in the problem context.

Enumeration refers to a student's quantification of units of volume. One key component in this thread is a student's strategy use for counting. Throughout the interview sessions we observed students utilize addition, skip-counting, and often multiplication for volume calculation.

For each student we identified what we call the moment of change in each of the four threads. In each thread, students demonstrated incremental developments toward more sophisticated understandings with instances of fallback. However, we identified a particular moment in the incremental development at which the student reached a level of understanding sufficient to produce accurate volume calculations and partial volume predictions. This does not imply that at that moment of change the student did, or even could, produce an accurate volume calculation or partial volume prediction. The moment of change only pertains to a single thread, but accurate volume calculations and partial volume predictions would only be produced consistently after the moment of change in all four threads. We describe the criteria for each moment of change in detail in section 4.5.1.

Having identified the moments of change for each student in all four threads, we explored patterns of growth and the interactions among the four threads. We found six patterns pertaining to students' growth across threads. These patterns are described in more detail in section 4.5.2. The patterns shed light on how students grow to understand volume and produce accurate volume calculations. In Sections 4.5.2.1-6 we share the stories of students who demonstrated each of the six patterns of growth. In sharing these stories, we discuss the interactions among the threads and key elements that led students to new understandings in each thread.

4 Results

4.1 An Overall Pattern of Growth

Throughout 11 tasks, the results demonstrate a pattern of growth among the students on both the paper portion of each task, and the portion of each task assisted by the dynamic environment. On the paper portion of the tasks the 31 students progressed from three students answering correctly on task 1 to 23 students answering correctly on task 10 and 21 students answering correctly on task 11. Figure A includes the total number of students providing the correct volume on the paper portion of each task.

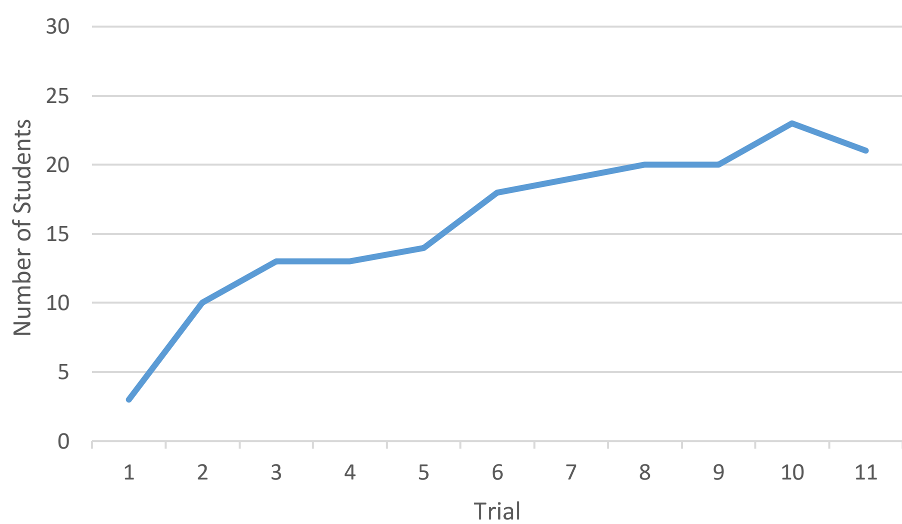


Figure A. *Total Number of Students Providing the Correct Volume on the Paper Portion of Each Task*

On the DVM portion of each task the students had the opportunity to make up to three correct predictions (prediction of the row, prediction of a layer, and prediction of the volume) while seeing a single cube. Thus, for 31 students there was the possibility of making a cumulative total of 93 correct predictions. Figure B includes the total number of correct predictions made by all students on each task while seeing a single cube. With the support of the dynamic environment, the students progressed from making a total of 28 correct predictions on task 1, to making a total of 75 correct predictions on task 10 and 72 correct predictions on task 11.

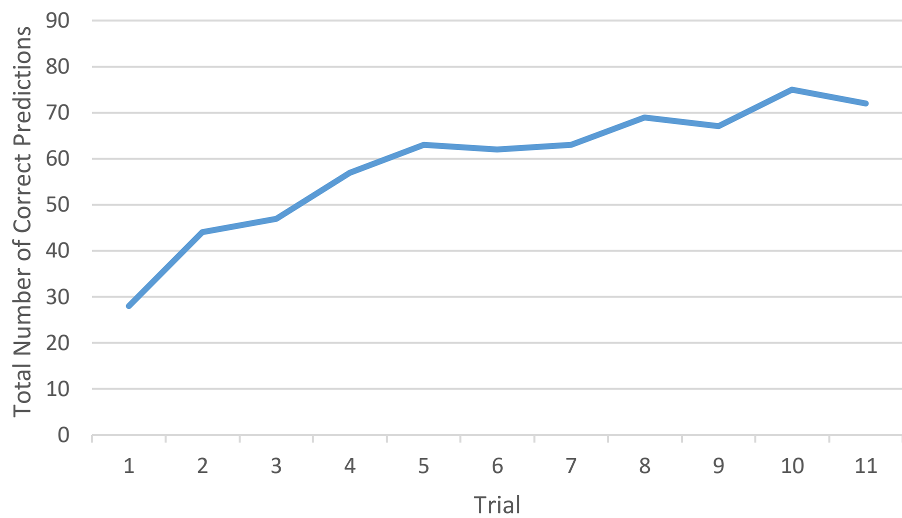


Figure B. *Total Number of Correct Predictions Students Made While Seeing a Single Cube on the DVM Portion of Each Task*

The improvement in the number of students providing correct volumes and predictions demonstrates the general effectiveness of the intervention. However, we also note that only 23 of 31 students provided correct volumes on the paper task at the peak. We interpret this to mean that several students may not have experienced benefits from the intervention or at least did not demonstrate empirically verifiable growth within the time frame of the intervention. Thus, we excluded the results of eight students who did not provide the correct volume on the paper portion of the task on any two consecutive tasks throughout the four interviews. In addition, these eight students did not demonstrate any consistent, appreciable improvement in their predictions throughout the tasks.

Of the remaining 23 students, we found six fourth graders and two third graders had prior knowledge that did not allow them to empirically demonstrate improvement in volume calculations or predictions throughout the tasks. Thus, we excluded the results of these eight students from the analysis. We analyzed the work of the remaining seven third graders and eight fourth graders who answered correctly on at least two consecutive tasks at least once within the four interviews and did not provide both correct predictions and volume calculations within the first two tasks. We describe these 15 students as those who demonstrated growth empirically. To summarize, we present findings from 15 students, excluding eight students who already knew how to measure volume and demonstrate that knowledge for prism measures, and we exclude eight other students who did not appear to learn the concept enough during our intervention to warrant any claims about how they learned it.

4.2

The first way that we looked for patterns in the growth of students' understanding ($n = 15$) of volume was to analyze the interactions of their work on the paper portion of each task and their work on the DVM portion of each task. The intent of the treatment was that the DVM portion of each task would support students in developing an approach to calculate volume through scaffolding volume unit counting, unit composition, and spatial structuring. Thus, we expected students to establish understandings and corresponding procedures on the DVM portion of the tasks, and transfer those understandings and procedures to the paper portion of subsequent tasks. However, the interactions among the two portions of each task were much more complex than we anticipated.

We note two general forms of interaction. We will illustrate the patterns with the type of chart shown in Figure C. The chart represents one student's responses on both the paper portion and the DVM portion of each task. We asked the student to make predictions related to the volume at several points within each task and each set of predictions is represented as a different series in the charts. When seeing only a cube on the paper portion of the task a correct volume prediction is represented as a one and an incorrect volume prediction is represented as a zero. When seeing only a cube on the DVM portion of the task each student had the opportunity to make three separate predictions (the number of cubes in a row, the number of cubes in a layer, and the number of cubes in the whole prism). Thus, we represent no correct predictions as zero, a correct prediction of only the number of cubes in a row as one third, correct predictions of the number of cubes in both a row and a layer as two thirds, and correct predictions of the number of cubes in a row, layer and the whole prism as one. Note that once a student made an incorrect prediction, we did not ask the remaining prediction questions when seeing only a cube. When seeing a row of cubes on the DVM we asked each student who had not correctly predicted the number of cubes in a layer to predict the number of cubes in a layer and the number of cubes in the whole prism. Thus, we represent no correct predictions as zero, a correct prediction of the number of cubes in a layer as one half, and correct predictions of both the number of cubes in a layer and in the whole prism as one. When seeing a layer of cubes on the DVM we asked each student who had not correctly predicted the number of cubes in the whole prism to predict the number of cubes in the whole prism. Thus, we represent an incorrect prediction as zero and a correct prediction as one.

Students demonstrating the first form of interaction treated the DVM and paper portions of the task as the same problem and attempted to take an integrated approach to solving the problem. By this we do not mean that they necessarily attempted to solve the two portions of each trial simultaneously. Instead we mean that students who took this integrated approach used strategies and results from one portion of the trial on the other portion of the trial early in the sequence of trials. These students generally gave their first correct volume on paper in the same trial that they made a full set of correct predictions. We found seven students who utilized this integrated approach. The results shown in Figure C for G4T4 typify the pattern of growth for students who utilized this integrated approach. We note that G4T4 made increasingly correct predictions on the DVM portion but did not produce a full set of correct predictions until trial 6. On trial 6, he produced a correct volume on paper before he demonstrated his ability to make a full set of predictions on the DVM portion. This indicates that when he had developed the ability to make correct predictions on the DVM portion he immediately applied it on the paper portion, demonstrating the integrated nature of his approach.

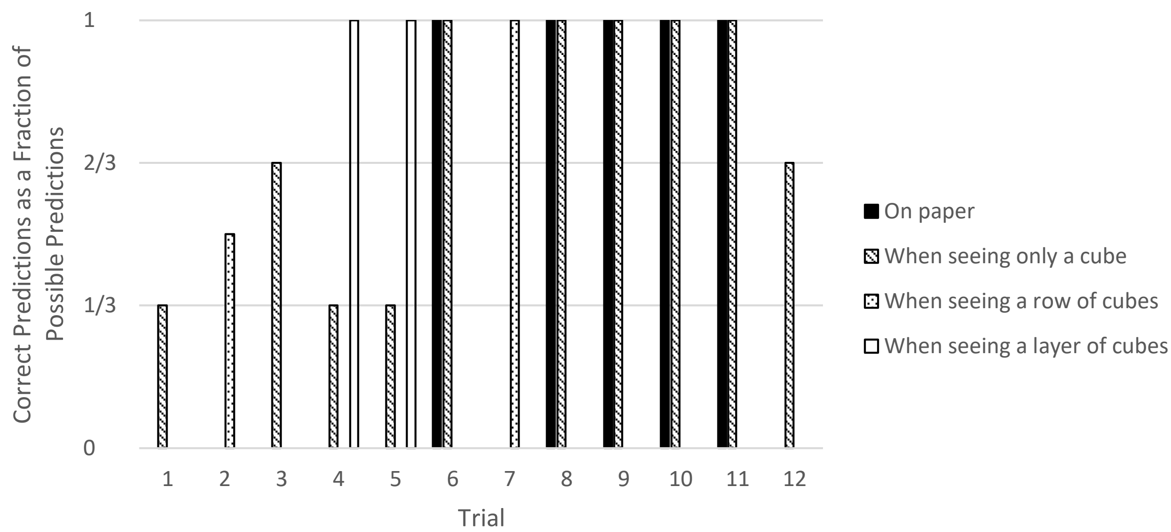


Figure C. *Correct Predictions G4T4 Made on Each Task as a Fraction of Possible Predictions*

The eight students who demonstrated the second form of interaction, treated the paper and DVM portions of each task as separate problems and attempted to solve each problem independently. This does not necessarily mean that students who took this independent approach did not recognize the two problems as analogous. Instead, some students may have allowed the independence as a measure of efficiency in problem solving, while others may have interpreted the two problems as independent problems.

In this form of interaction, two students developed an appropriate strategy for volume calculation on the paper portion of the tasks prior to developing an accurate strategy to predict the partial volumes required for the DVM portion of the tasks. Figure D shows G3J3's responses, which demonstrate how her correct predictions on the DVM portion followed her correct responses on the paper portion. We also note that her progression of predictions on the DVM portion was still incremental despite that she had already established an appropriate algorithm to calculate volume on the paper portion.

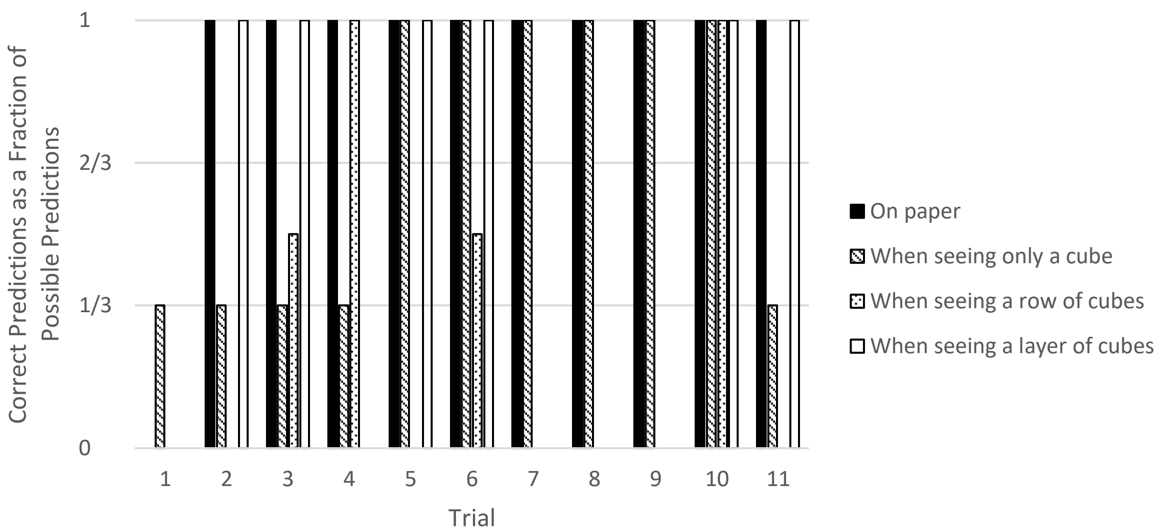


Figure D. *Correct Predictions G3J3 Made on Each Task as a Fraction of Possible Predictions*

In the same form of interaction, six students developed an appropriate strategy for volume calculation on the paper portion only after developing an accurate strategy to predict the partial volumes required for the DVM portion of the tasks. These students tended to show incremental improvements on the predictions for the DVM portion leading to an accurate prediction strategy. Then, the students applied the strategy developed for predicting to their volume calculations on the paper portion of subsequent tasks. When G4C2 first provided a correct volume on the paper portion he described his process saying, "I tried to use my computer strategy." Figure E, showing G4C2's responses, demonstrates the incremental improvements in predictions and the lag in application to the paper portion. We note that he made a full set of correct predictions on several tasks prior to transferring that strategy to his work on the paper portion and determining a correct volume. Some students applied the strategy quickly, while others completed as many as six tasks before applying the prediction strategy on paper.

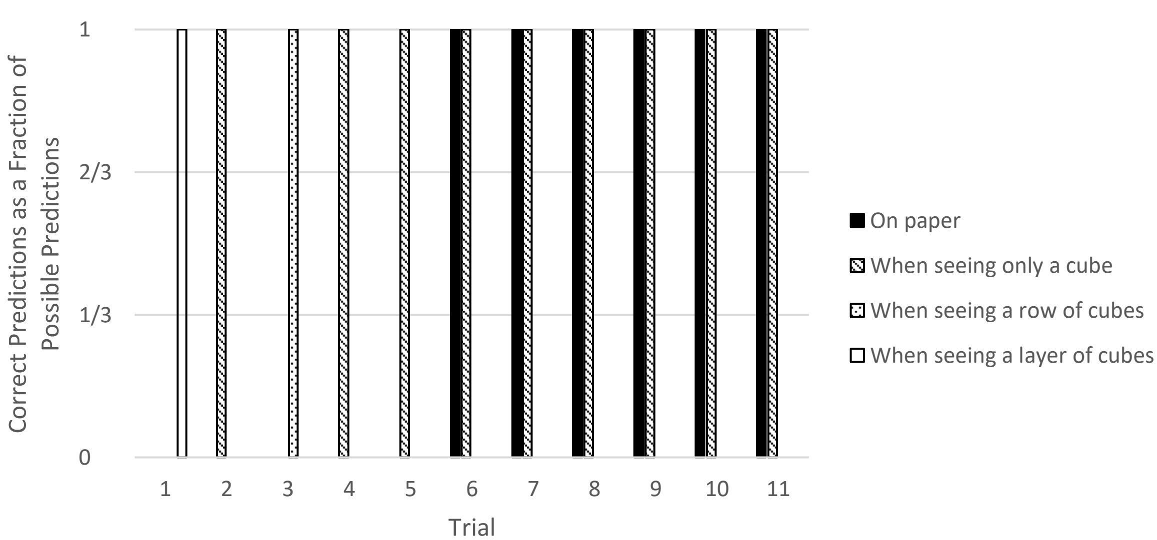


Figure E. *Correct Predictions G4C2 Made on Each Task as a Fraction of Possible Predictions*

4.4 Level progressions

The second way that we looked for patterns in the growth of students' understanding of volume was to identify students' progression through the levels of an LT for volume (Barrett, etal 2017). We found that the students who demonstrated growth progressed from the VURR level or below to the VRCS level of the LT. This demonstrates the efficacy of the intervention to produce measurable growth in students' understanding of volume. One student (G3L2) demonstrated VRCS level thinking only on the paper portion of the tasks, and not on the DVM portion. Table AA (Appendix A) shows each student's level progression across the 12 tasks.

Looking closely at student work in the tasks just before and just after a level change occurred, we found additional aspects of growth that are not adequately captured by a level-based analysis. Thus, we utilized a third approach to identify patterns of growth. In section 4.5 we describe the results from this third perspective.

4.5 The Four Threads of Growth

The third way that we looked for patterns in the growth of students' understanding of volume was to analyze the interactions among students' four fundamental threads of growth we refer to as interpretation, representation, structure, and enumeration. These threads interact throughout the development of students' understanding of strategies to determine volume. For a thorough description of the threads, refer to section x.

4.5.1 Moments of Change

In the interpretation thread, we noted the moment of change once students interpreted the question of volume in terms of cubes filling space, and recognized the DVM assisted volume predictions and the volume calculation on paper as the same questions. In every case, the DVM assisted volume predictions came first so that the volume calculation on paper was always used to identify the moment of change. This follows because we claim that interpretation requires the transfer of a conceptual account of cubes filling up space that would transcend the context of paper drawings of prisms or DVM-drawn prisms. The moment of change in the interpretation thread occurred anywhere from task 3 to task 11, with one student who never interpreted the questions as analogous. Excluding the student who did not reach this interpretation, the mean task was 5.6, the median task was 5.5, and the sample standard deviation was 2.6.

In the representation thread, we noted the moment of change once students utilized the length labels to determine the number of unit cubes along an edge of the prism. Other aspects of the representation thread, such as recognizing the three-dimensional structure of the two-dimensional paper representation, often preceded this understanding. The appropriate utilization of the labels was the most empirically verifiable element necessary for correct calculations and predictions in this thread, so we used this as the indicator of the moment of change. The moment of change in the representation thread occurred anywhere from task 2 to task 10. The mean task for the moment of change was task 4.7, the median was task 5, and the sample standard deviation was 2.3.

In the structure thread, we noted the moment of change once students noted the use of composite units to form rows, layers, and a volume. Here we see structure as a way of coordinating groups into constituent units or collecting and combining existing groups to form a larger group that can be counted meaningfully, without losing the reference to the smaller unit groups and singular units that it contains. The moment of change in the structure thread occurred anywhere from task 2 to task 8. The mean task for the moment of change was task 3.9, the median task was 4, and the sample standard deviation was 1.5.

In the enumeration thread, we noted the moment of change once students utilized multiplication effectively within their calculations or predictions. This moment of change is unique because it was not a necessary condition for correct predictions or volume calculation. We chose this moment of change because every student in the study demonstrated the necessary addition skills to produce accurate volume calculations with varying degrees of efficiency. The moment of change in the structure thread occurred anywhere from task 1 to task 8. The mean task for the moment of change was task 4.3, the median was task 4, and sample standard deviation was 2.3.

We focus our analysis of the threads primarily on the threads interpretation, representation, and structure. Our minimal analysis of the enumeration thread can be attributed to two factors. First, the unique moment of change in the enumeration thread. Second, the protocol we enacted for this study constrained the data we collected on enumeration. Our protocol took the focus off students' computation abilities by allowing the students to ask for assistance with computation. In addition, we focused our probing questions around other aspects of students' strategies and not around computational methods. This decision allowed us to investigate other aspects of student strategy use deeply, but may have limited our findings in the area of enumeration. However, we note two informal results about enumeration in section 4.7.3.

4.5.2 Patterns in the Moments of Change

As we identified moments of change in the three primary threads, we found six different patterns concerning the order of development. Some students reached the moment of change in all three threads within one task, while others reached the moment of change in each thread on different tasks. In addition, some students reached the moment of change in the structure thread before the others, while some students reached the moment of change in the interpretation or representation thread before the others. Table A summarizes the sequential patterns of growth in relation to the three threads. Each pattern is notated as a hyphenated list of the three threads in the order (from left to right) of the moments of change in each thread. Any threads listed in parentheses had the moment of change occur on the same task. The table also includes when the students started using multiplication as the moment of change for the enumeration thread.

Table A

Patterns of Growth Demonstrated in the Four Threads

Pattern	Number of Students
Structure-Interpretation-Representation	
Enumeration between Interpretation & Representation	1
(Structure-Interpretation)-Representation	

Enumeration before Structure	1
Enumeration between Interpretation and Representation	1
(Structure-Representation)-Interpretation	
Enumeration before Structure	1
Enumeration between Representation and Interpretation	5
Interpretation-(Structure-Representation)	
Enumeration same as Interpretation	1
(Structure-Representation-Interpretation)	
Enumeration before Structure	2
Enumeration same as Structure	2
Structure-Representation	1

In the following sections we provide a brief narrative of the growth of one of the students from each pattern listed in Table A. In these narratives we focus on a few elements of development including the interactions among the threads of growth, students' mental structuring of three-dimensional space, and circumstances that may have motivated the moment of change in each thread.

4.5.2.1 Structure-Interpretation-Representation

G3K1 was the only student who demonstrated the Structure-Interpretation-Representation pattern of growth. She reached the moment of change in structure on task 4 but used her understanding of structure only for predictions. On task 7 she reached the moment of change in the interpretation thread when she used the structuring on paper. Her moment of change in the representation thread came on task 8 when she connected the length labels to the number of cubes along an edge.

On the paper portion of task 1, G3K1 added all the reported edge lengths. On the DVM portion, she did not make any correct predictions. The researcher asked her if the numbers could help her. Although she answered yes, G3K1 used strictly imagistic reasoning to approximate the number of cubes from task 2 until task 7.

On task 2, G3K1 drew a line from the cube to the rectangular prism (see Figure F). Then, she said the volume was 60 (incorrect) and explained that she pictured putting the cubes in the box in her mind. On the DVM portion, G3K1 correctly predicted the number of cubes that would make a row, but not the number of cubes that would make a layer. Once she saw the bottom layer of the rectangular prism, she predicted the total volume correctly. G3K1 correctly predicted the total volume once she saw the bottom layer on the DVM on every subsequent task. This demonstrates the incremental development of her mental structuring of three-dimensional space.

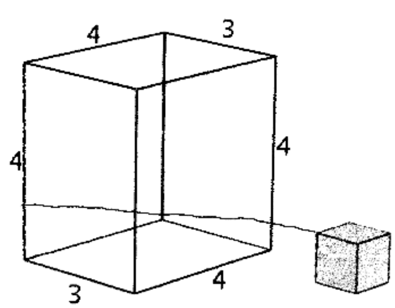


Figure F. *G3K1's Written Work on the Paper Portion of Task 2*

On the next task, G3K1 said the volume was 28 (incorrect) and stated that she guessed. On the DVM portion, she incorrectly predicted the number of cubes in a row and in a layer. These, and subsequent incorrect predictions demonstrate that G3K1 was not using the length labels, but a visualization strategy, to estimate how many cubes fit along a given edge. However, her consistent, correct predictions when seeing a layer show that her visual estimation strategy was relatively effective, particularly in determining the height of the prism. This may have contributed to the retention of her visual estimation strategy through task 8.

G3K1 drew horizontal lines across the figure approximately the height of the cube apart on task 4 (see Figure G). She also drew vertical lines and pointed to each square in the grid, which was 30 squares. Then, she said the answer was, "maybe 20." On the DVM portion, G3K1 made correct predictions for the row, layer, and total volume. We take these predictions as evidence that she had established an adequate mental spatial structuring scheme at this point, indicating a moment of change in the structure thread. We attribute subsequent, incorrect predictions to her continued use of visual estimation in place of length labels to determine how many cubes fit along each edge, rather than a lack of structuring.

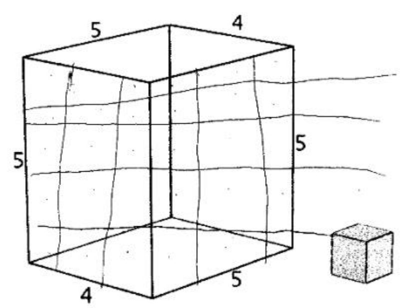


Figure G. *G3K1's Written Work on the Paper Portion of Task 4*

On task 5, G3K1 drew on the prism (see Figure H). Then, she counted seven squares on what appears to be a bottom layer because she only counted the two squares at the corner once. We surmise that she counted the two squares as one because she knew they represented the two faces of the corner cube, but we note that she may have been imagining a plane and ignoring the pre-drawn line through her plane. She counted the number of squares vertically (four) and multiplied the values (7 x 4) to report an incorrect volume of 28. On the DVM portion of task 5, G3K1 correctly predicted the number of cubes in a row (three). She incorrectly predicted 12 cubes would make a layer, explaining that there would be four rows. Later, when seeing a layer of 18, she added three 18s together to predict the correct volume. We take her description of rows and prediction using layers as further evidence that her mental structuring scheme was established, although her visual estimation of length was not adequate to make the appropriate predictions.

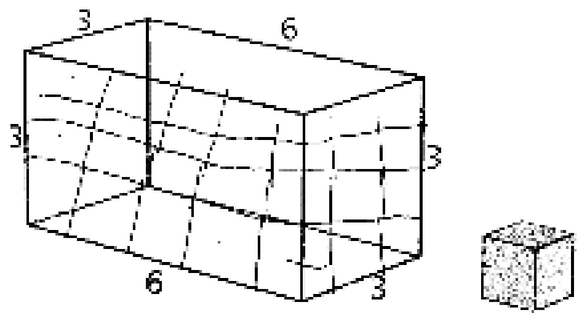


Figure H. *G3K1's Written Work on the Paper Portion of Task 5*

When the bottom layer was complete on the DVM, the researcher asked, "Is there a way you could have done it differently?" She said, "Yeah… Let's see, there is a six here and three here. Oh, I see." After filling the prism on the DVM G3K1 said, "Maybe when I do this [pointing to her paper] it isn't so helpful." The researcher asked, "Is there another way you could try it?" She replied, "Maybe do times and stuff." Then the researcher asked G3K1 if she would still know the volume would be 54 if the DVM did not show the volume. G3K1 said, "Yeah, I would count it… I would memorize the green equals row, and this equals more rows, and this equals to take what was on the bottom and copy it on top, so I would maybe do that in my mind." The conversation on this task may have been instrumental in G3K1's growth in the interpretation, representation, and enumeration threads. Although she did not immediately demonstrate growth, she made significant changes in her strategies that led to correct calculations and predictions within two tasks.

On Task 6, G3K1 drew on the prism as shown in Figure I. She computed the products of 5 times 6 and 2 times 2 and added 30, 30, and 4. We attribute her use of the length labels and multiplication to the conversation on the previous task when she noted that the labels may be important and that multiplication could be helpful. On the DVM portion, G3K1 correctly predicted the number of cubes in a row, but not the number of cubes in a layer. We note that although she used the length labels on the paper portion, she still did not recognize their utility for predicting on the DVM portion of the task.

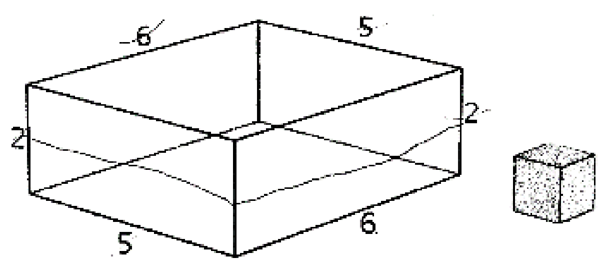


Figure I. *G3K1's Written Work on the Paper Portion of Task 6*

On the paper portion of Task 7, G3K1 drew on the bottom face and the back right face of the prism, but did not match her partitioning with the length labels (see Figure J). She said, "I looked at the bottom there would be 1, 2, 3, 4, 5 and then there's three rows of that… that would be 15. And then I counted it up here and that would probably be four. And then 15, there's four 15s and that equals 60." This is the first time that G3K1 described the use of this type of structuring on the paper portion of the task. We take this as evidence that she has reinterpreted the problem as analogous to the predictions on the DVM portion and we classify this as her moment of change in the interpretation thread. During this task, G3K1 said, "Sometimes I don't draw it right." This was the last task she attempted a drawing on paper. Note that she recognized the incorrect aspects of her drawing, despite finding the correct volume. This is evidence that she is attending to the entire structuring process. On this task G3K1 mentioned, "it kind of gives you clues," referring to the length labels. We take this statement as evidence that she recognized the utility of the length labels for her calculations.

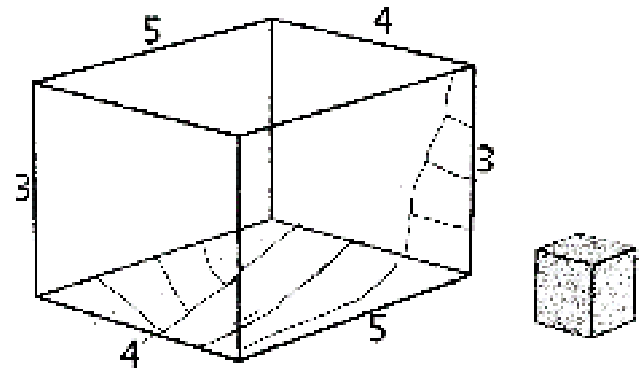


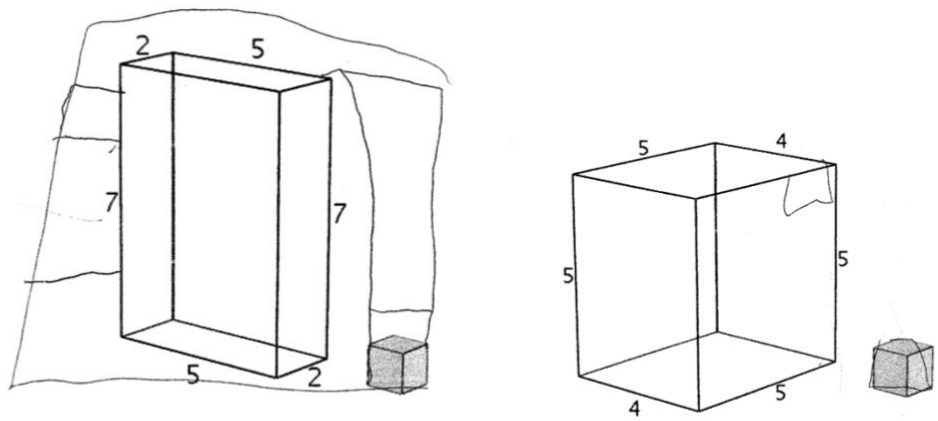
Figure J. *G3K1's Written Work on the Paper Portion of Task 7*

On Task 8, G3K1 calculated the volume correctly on the paper portion and made correct predictions on the DVM portion. She, again, explained that the numbers give her clues. Because G3K1 demonstrated the appropriate use the length labels on this task, we classify this as her moment of change for the representation thread. She continued to produce correct volume calculations and predictions through the remainder of the tasks and her confidence and excitement about doing so were evident.

4.5.2.2 (Structure-Interpretation)-Representation

The two students who demonstrated this pattern of change were very different in many other ways. One student experienced the moment of change in structure and interpretation on task 3, followed by representation on task 6. The other student did not experience the moment of change on structure and interpretation until task 7, followed by representation on task 10. We recount the narrative of this second student because the prolonged struggle helps reveal more detail about the growth of understanding.

On the first four tasks of the paper version G3M1 stated that she guessed to find the answer. However, she produced drawings on task 1 and task 4 that indicated she may have been using some visual measuring with the small cube (see Figure K). On these four tasks she predicted the number of cubes in a row correctly twice. On the other two tasks, her prediction for a row would have been correct for the adjacent edge length on the base of the prism. This suggests she may have been using the length labels to make these predictions, but she also stated that the labels were not helpful when asked about them on task 4. This may have been because her predictions were still incorrect when she utilized the labels. This negative feedback may have caused her to avoid the labels for an extended time. Also on task 4, the interviewer asked if she could have figured out the volume before using the DVM. She described guessing the number of cubes in a row, layer, and the volume similar to the DVM process. This indicates she may have been developing spatial structuring to mimic the DVM's structuring scheme, although she was unable to coordinate it appropriately for use until later tasks.



Task 1 Task 4

Figure K. *G3M1's Written Work on the Paper Portion of Task 1 and 4*

On task 5, G3M1 used her fingers as a form of measurement to determine the number of cubes in the rectangular prism. She held two fingers over the cube pictured on the paper because her two fingers provided an approximation of the size of the cube. She then iterated her fingers on the representation of the prism. It appeared she iterated her measure along the three colored edges of the prism, but it was unclear how she produced her overall volume of 60 from her measurements. We think the question posed by the researcher on task 4 may have been instrumental in prompting this change in strategy. On the DVM portion she used a single finger in a similar manner to make predictions because a single finger was a better approximation of the size of the cube on the DVM. She did not predict a row correctly when seeing a cube, but adjusted how she used her finger and when seeing a row she predicted the layer and volume correctly. This is further evidence that she was beginning to appropriately structure space to make predictions but did not yet understand the length labels as indicating the number of volume units fitting along a side.

G3M1's performance and strategies were much the same on task 6 as on task 5. She used two fingers to measure on the paper portion and one finger to measure on the DVM portion. After her prediction for a row on the DVM portion, the interviewer asked if the numbers would help her and she said no. On this task she incorrectly predicted a row, but correctly predicted a layer and volume when seeing a row. After seeing a layer, the interviewer asked her to predict the volume again. She used the idea of rows to recount how many cubes were in a layer before making a correct prediction for volume. This indicates that she was using structural thinking, but still struggled to coordinate units of units into composite units.

On the seventh task on the paper version, G3M1 explained that she found how many cubes fit along the two edges of the base using her fingers to measure and she multiplied those values. Then, she measured how many cubes tall the prism was and multiplied that with her previous product to find the volume. Her explanation indicated that she had developed a coordinated spatial structuring that we take as evidence of her moment of change in the structure thread. However, she was still measuring the lengths using her two-finger strategy. On the DVM portion she appeared to use her prior calculations from the paper portion because she did not take time to make any calculations and did not use her single finger strategy to estimate but responded immediately with predictions. This indicates that she had also coordinated the two problems and reached the moment of change in the interpretation thread.

For the paper portion of tasks 8 and 9 G3M1 used the same strategy as on task 7, including her two-finger strategy. On the DVM portion she continued to use her predictions from the paper portion of each task. When checking her predictions on the DVM portion of task 9 G3M1 said she wanted to build outside the prism. She built the rectangular prism to fit her predictions. When she was asked to put it inside to see if it fit, she saw that it did not fit. She then returned the image to a single cube and predicted each step correctly while seeing a single cube. We believe at this point she used the numbers on the page instead of her finger measuring technique.

After this task she got the DVM and the paper portions correct for the rest of the tasks. We believe that building according to her predictions and then seeing the mismatch was instrumental in building her connection of the length labels with the number of cubes that would fit along an edge. She had repeatedly denied the utility of the length labels up to this point, perhaps because her prior predictions based on the labels were incorrect. Building to fill the prism may have somehow masked the disconnect between her length estimations and the actual lengths so that she did not recognize the length labels as a better predictor than her physical measurement strategies.

4.5.2.3 (Structure-Representation)-Interpretation

Among the six students who demonstrated this pattern of growth, we note two distinctions. For three of the students, the moment of change in the interpretation thread followed quickly after the moment of change in the other two threads. For the other three students there was more significant lag time after the moment of change in the first two threads. The three students who showed this lag time all used surface areas strategies on the paper portion of the task. We return to the analysis of this lag time in section 4.7.2. However, we focus on the case of G4T1, who did not show lag time, for this section. Her moment of change for structure and representation came on task 5 and interpretation on task 6.

On the first task, G4T1 drew nearly correct tilings on the two visible faces of the prism (see Figure L). However, it is unclear how she conceptualized the tiling in terms of volume because she said the volume was “maybe 15.” She did not make any correct predictions but did use a multiple of a visible row to predict the layer and noted her prediction was incorrect because she multiplied 3 times 3 instead of 3 times 5.

On task 2, she calculated the area of the two visible, vertical faces correctly with multiplication and added the products to predict the volume on the paper portion of the task. She correctly predicted a row, and again used a multiple of the row to predict a layer. She also used a multiple of a layer to predict the volume when seeing only part of a layer. These results indicate that G4T1 was already developing spatial structuring schemes, but the schemes were different for the paper and DVM portions of the task.

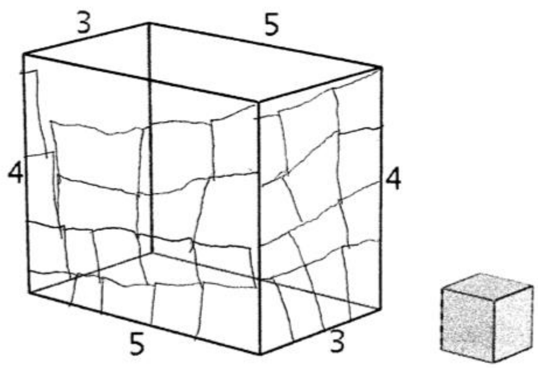


Figure L. *G4T1's Written Work on the Paper Portion of Task 1*

On task 3, G4T1 moved her pen around the prism in what appeared to be a purposeful row-by-row structure and stated she was trying to "imagine the cubes in the box and trying to count it." This move signifies her attention to volume as counting cubes and demonstrates the shift in her reasoning from surface area strategies to three-dimensional counting strategies on the paper portion of the task. We do not interpret this as the moment of change in the interpretation thread because she does not demonstrate that she views the DVM and paper portions as identical. However, this does represent a significant shift from interpreting the volume question two-dimensionally to three-dimensionally. We believe that this shift was influenced by her work in the DVM. On the DVM portion of the task she predicted a row and layer correctly and used a multiple of the layer to predict the volume, all while seeing only a single cube.

Tasks 4 and 5 mirrored tasks 1 and 2 for the paper portion of the tasks. On task 4, G4T1 returned to the same strategy she used on task 1, drawing on the two visible, vertical faces of the prism and counting the tiling, but she counted the tiling correctly. However, she showed skepticism in her strategy saying, "[I am] not really sure if this is correct." With the DVM, she predicted a row and layer correctly seeing only a cube but added the vertical dimension to her layer prediction as a prediction for volume. On task 5, she used the sum of the areas of the two visible, vertical faces, which she found correctly by multiplication. Then, she predicted everything correctly on the DVM portion. We take these correct predictions as evidence of her moment of change in the structure and representation threads because she continued to make correct predictions on subsequent tasks.

On task 6, G4T1 described the whole bottom of the box as 25 and added three 25s for the volume. Although a layer was 30 cubes (six rows of five), we take this as evidence that she was attempting to utilize the structuring she had learned through the use of the DVM on the paper because her strategy was significantly different than on previous tasks. We identify this as the moment of change in the interpretation thread. On the DVM portion, she predicted a row correctly, but predicted a layer as 25. This provides further evidence that she was interpreting the paper portion as the same problem because she used the value from the paper on the DVM. Similarly, on task 7, she used the same incorrect calculation on the paper and DVM portions of the task. She used correct row and layer predictions, but she did not predict the whole volume correctly. After using the DVM on task 7, she explained what she had done incorrectly in her calculations and proceeded to do everything correctly on the remainder of the tasks.

Task 6 and 7 present an interesting scenario in the context of our study. G4T1 appeared to have reached an adequate understanding in all three threads by the beginning of task 6 to make correct predictions and volume calculations on both the paper and DVM portions of the tasks. However, she failed to find the correct volume on those tasks and transferred her incorrect calculations to the DVM, a place she had successfully calculated volume before. This demonstrates the complexity of integrating these threads for calculations in a single problem context. Although G4T1 demonstrated adequate understanding in each thread, she struggled to utilize all three simultaneously to effectively approach the problem on paper. We found that this is the same struggle experienced by those in this category who had greater lag time before their interpretation moment of change and will be discussed in greater depth in section 4.7.2.

4.5.2.4 Interpretation-(Structure-Representation)

Only one student fit this pattern of change. G4J1 showed early signs of structural reasoning and reached the moment of change in the interpretation thread on task 3. However, a misinterpretation of the DVM influenced his reasoning about structure and representation through task 3 and he reached a moment of change in those threads on task 4.

G4J1 began work on task 1 by correctly determining there should be five cubes along one edge of the base. He then counted by fives while moving his finger up the prism. His skip counting led to an incorrect volume calculation of 25. Here, he already demonstrated a focus on using cubes to find volume. He also showed that he conceptualized a structuring scheme involving groups of units. On the DVM portion, he counted while moving his curser along the bottom of the prism, pausing every two units, "one, 2… 3, 4… 5, 6… 7, 8… 9, 10." The bottom of the prism was 2 units by 5 units, so we take this as further evidence of a reasonable spatial structuring scheme. Furthermore, he skip-counted by tens and claimed the volume was 50, not the 25 he had earlier predicted. Although this volume was incorrect (40 not 50), it appears that he had strong mental spatial structuring skills. However, we did not consider this sufficient evidence to claim he began the task with an adequate understanding of the structuring thread because he did not utilize this type of structuring consistently until later in the tasks. G4J1 predicted a row correctly but did not predict a layer correctly until the row was present. As he worked with the DVM, he continually readjusted his interpretation of how the program operated. When he started using the red button to iterate layers he settled on an interpretation that each push of a button doubled the amount of cubes or "did times 2 every time."

On task 2, G4J1 drew a cube inside the box. Counting four along one edge of the base and then moving his pen along the adjacent edge he said, "four, five is 20." We interpret this as multiplication to determine the number of cubes in a layer. Then, he moved his pen along a vertical edge and said, "Twenty times 3." This would have given the correct volume, but then he said, "Twenty times 2. Forty. I think it's 40. Forty cubic units that go all inside of it." We believe he changed his strategy to match his conception of how the DVM operated (using doubling), but it is also possible that he changed his determination of the height of the prism. Assuming his calculation was based on his doubling strategy we do not identify this as the moment of change in either the structure or interpretation threads. Although there is evidence G4J1 had an adequate spatial structuring scheme, he gave it up in favor of his doubling strategy, demonstrating that his spatial structuring scheme was not strong enough to stand in the face of opposing ideas. In the same sense, his use of the length label as indicating the number of layers in the prism was not strong enough to stand in the face of this opposing idea. On the DVM portion he predicted a row and layer correctly but did not predict the volume correctly while seeing a cube or at any other point. Although he appeared to use his understanding of the DVM to change his prediction on paper, his DVM prediction for the whole volume was 30, and not the 40 he calculated on paper. Therefore, we do not interpret this as the moment of change in the interpretation thread.

On task 3, G4J1 determined there “would be nine in the bottom,” which is the correct number of cubes to form a layer. Then, he added three nines together, then added 27 and 27 for the volume. If he had added four nines he would have calculated the correct volume. We were not able to determine if he mistakenly added 27 when he meant to add another nine, if he reverted to his doubling strategy, or had other reasoning to support his calculation. Therefore, we still withhold labeling this the moment of change in either the structure or representation threads. On the DVM portion, he once again predicted the row and layer correctly, but used his incorrect calculation from the paper to predict the volume. We take this as evidence of the moment of change in the interpretation thread.

On task 4, G4J1 explained that he was “using what the computer does” and calculated the correct volume. He also made all correct predictions on the DVM portion and continued to calculate and predict volume correctly in subsequent tasks. We take this as evidence that he reached the moment of change for the structure and representation threads on task 4.

G4J1 presents a challenging case for pinpointing the moment of change in the representation and structure threads. Even on task 1, he demonstrated strong spatial structuring skills. On task 2, we could argue that he reached the moment of change in the structuring thread and in the representation thread because he presented all the appropriate structural elements with the appropriate measurements prior to adjusting his calculation from 20 times 3 to 20 times 2. However, we made the case that this did not represent the moment of change because neither understanding was strong enough to be retained against the opposing idea of his doubling strategy. Interestingly, this doubling strategy was developed on the DVM portion of the task and he utilized it in this case on the paper portion. Thus, it appears that he is interpreting the two questions as analogous. Although this indicates he may have reached the moment of change in the interpretation thread, we did not classify this as such because he did not utilize his paper calculations to inform his DVM predictions until task 3. G4J1’s reasoning demonstrates the complex interactions among these three threads. Although he had nearly reached the moment of change in all three threads, the coordination of the threads held him back from the final solidification of his ideas because he had misinterpreted the DVM’s actions.

4.5.2.5 (Structure -Representation-Interpretation)

Four students followed this pattern of growth. The cases are very similar in many respects. The growth happened relatively quickly for G4C4, so we have chosen not to share her story. G3J3 was a unique case in which she happened to acquire the appropriate procedures for volume calculation on task 2 through purposed guessing. However, she did not reach the moment of change until task 6 in any of the three threads. She did not understand volume calculation, only a detached algorithm for calculation. The ways she learned within the threads were similar to others, but we do not share her story in order that we do not distract with the unique aspects of her story. In the interviews for G3C3 there were some small deviations from the protocol that, although they do not invalidate any results, detract from the consistency of this presentation. Therefore, we share the story of G3C1. She reached the moment of change for all three threads on task 6.

On task 1, G3C1 guessed three on the paper portion. Then, she predicted a row correctly and a layer incorrectly. However, she used a multiple of the row to predict the layer. When seeing a layer she predicted a multiple of the layer and explained how she obtained her prediction as a sum of 15s (the amount in a layer). When asked about the true volume she said, "isn't it [paper] the same as this one [DVM]?" This would seem to indicate she was close to a moment of change in the interpretation thread, but something kept her from the moment of change until task 6. We recognize that it may have been asking the question, "What is the actual volume?" that kept her from acknowledging the analogous problems.

G3C1 said, "it looks like there may be ten on the first one and another ten on the top row" on the paper portion of task 2 when asked to explain her answer of 20. On the DVM portion, she predicted a row correctly as six cubes. After the interviewer asked her how she thought about her initial predictions for a layer, she said, "actually 12 because 6 plus 6 would be 12." After seeing a row and retaining her prediction, the interviewer spun the prism and she changed her prediction to 18, which was correct. After seeing a layer, she predicted the volume correctly.

On task 3, G3C1 said her prediction of 42 on the paper portion was "just a little guess." She predicted a row correctly, but not a layer. She did not initially use a multiple of the row for her layer prediction but predicted a number close to the actual layer. She may have simply been using visual estimation skills by comparing with the previous prisms. After having a layer, she predicted the volume correctly.

For the paper portion of task 4, G3C1 clarified, "So, I can just guess." The interviewer said, "You can guess, or you can tell me what you think it is by thinking about this," as he pointed to the figure on the paper. She incorrectly guessed 50. On the DVM portion, she predicted a row correctly and used a multiple of a row to predict the layer when seeing only a cube. When seeing a layer her volume prediction was a multiple of the layer. After filling the prism, the interviewer asked if she could tell the volume was 40 without the volume label on the DVM. She explained, "Yeah, because it's easy, it goes by 10s. Ten, 20, 30, 40."

On task 5, G3C1 wrote $9 + 9 + 9 + 9 + 9 = 45$ along the base of the prism and predicted the volume was 45. She did not verbalize that she was thinking about walls, but a wall in the prism would have been 9 cubes and six of those layers would have filled the prism. On the DVM portion she predicted both a row and layer correctly. However, she predicted too many layers until after she saw a layer and changed her prediction to the correct volume. We could argue that she reached the moment of change in the structuring thread on this task. However, the discrepancy between her work on the paper and DVM is evidence that she may not have fully settled on a structuring scheme until the very end of task 5 or the beginning of task 6.

On task 6, G3C1 calculated the correct volume on the paper portion writing 5 x 6 = 30 x 3 = 90. Her initial prediction for a row on the DVM portion was six, when it should have been five. However, her prediction for a layer was 30 and for the volume was 90, both of which were correct. This indicates she had switched the two horizontal edge lengths when making her predictions. She noticed this once she saw a row of cubes. Although she switched the edge lengths, we take her accurate volume calculation on the paper portion, and primarily correct predictions as evidence that she had reached the moment of change in all three threads by the end of task 6. This is supported by her correct calculations and predictions on subsequent tasks.

4.5.2.6 Structure-Representation

One student who demonstrated growth, did not reach a defined moment of change in the interpretation thread. On task 1, G3L2 drew three-dimensional cubes in the prism on paper using visual perception to estimate how many would fit along an edge. She struggled to draw and keep track of her cubes beyond a single layer and ended up estimating the volume incorrectly. She did not make any predictions on the first task.

On task 2, she drew squares on the base of the prism instead of attempting a three-dimensional representation. On the DVM portion she predicted a row correctly and used a multiple of the row to predict a layer, but she did not make a prediction for the volume.

On task 3, she drew squares on the front face of the prism that matched the dimensions of the face. Then, she added the number of squares on the front face together several times and predicted the correct volume, indicating she was thinking about vertical layers in the prism. Because she continued to use a similar pattern throughout all but one of the subsequent tasks, we label this the moment of change in the structure thread. However, it is unclear if she was using the length labels for her predictions or if her visual estimation matched the actual dimensions. Because she did not continue to use the length labels, we do not label this the moment of change in the representation thread. On the DVM portion of this task she did not make correct predictions and seemed surprised at the actions of the DVM. In particular, when pushing the button to iterate a row, she said she expected it to “go up,” instead of repeating the row horizontally.

On task 4, G3L2 drew a tiling on the back, left face of the prism that did not match the length labels and added the number of squares together according to her length estimation for the perpendicular dimension. On the DVM portion of task 4 she did not make any predictions because she did not remember what the buttons did.

On task 5, G3L2 added together all of the numbers on the diagram instead of using the structuring scheme she had previously utilized on tasks 3 and 4. We believe she released the structuring scheme momentarily because she did not produce a correct answer on task 4 because she did not utilize the length labels. This illustrates the interaction among the representation and structure threads. On the DVM portion she correctly predicted a row and used a multiple of the row to predict a layer but was still perplexed that the DVM iterated rows horizontally instead of vertically.

G3L2 returned to drawing a tiling on the back, left face of the prism on the paper portion of task 6. She used repeated addition and explained that she used six addends because, “on here it says six,” pointing to the appropriate length label. This indicates that she was utilizing the length labels to indicate the number of cubes along an edge. This is her moment of change in the representation thread because she continued to utilize the length labels appropriately on paper. On the DVM portion, she made correct predictions about what she would do (e.g. push the button twice) but claimed she did not know how many cubes there would be and did not expect her predictions to be correct.

From task 6 to task 9 she calculated the volume correctly on paper using her vertical layer structuring strategy and appropriate length labels. Although she did not make correct predictions, we attribute this to a lack of coordination among her spatial structuring scheme and that of the DVM. We label this as a struggle in the interpretation thread because she did not translate between the representations as analogous structures.

On tasks 10 and 11, G3L2 did not calculate the volume correctly on paper. She drew a tiling just as in previous tasks, but she used only one dimension of the tiling instead of the area to calculate the volume. We attribute this to the interaction of three elements. First, we suspect that because she had not adequately integrated her structuring and that of the DVM, she had not solidified her structuring scheme for long term retention. Second, the break between her session with tasks 7 to 9 and her session with tasks 10 and 11 was longer than other students because she was ill. Third, she began utilizing multiplication as opposed to repeated addition on task 7 at the suggestion of an interviewer and we believe she may not have appropriately integrated her use of multiplication with her structuring scheme and misapplied it in tasks 10 and 11. We believe each of these three factors played a role in her fallback on tasks 10 and 11.

4.7 Exceptionalities

Among the students who demonstrated growth, we found significant variation in the task number when a student experienced a moment of change in a given thread. Although variation is to be expected, we found students for each of the four threads who demonstrated a prolonged struggle in that particular thread. In each case the student seemed to be on the cusp of understanding but could not tip themselves into the moment of change for several tasks. In the following sections we recount the challenges that we found held the students at bay and the elements that led to their eventual moment of change.

4.7.1 Exceptionalities in the Representation Thread

In the representation thread the interpretation of the length labels as indicating the number of cubes that would fit along an edge was particularly challenging for G3K1 and G3M1. We recounted G3K1's story in section 4.5.2.1 and G3M1's story in section 4.5.2.2. G3K1 used the length labels initially on task 1, but her volume calculation on the paper portion was incorrect and her prediction for a layer on the DVM portion was incorrect. At the end of task 1 the researcher asked if the numbers were helpful and G3K1 said yes. However, she may have interpreted the interviewer's question as hinting that she should not use the length labels because when she had used them she was incorrect. After task 1, she used visual estimation strategies until task 6. On task 6 she used the length labels but calculated the volume incorrectly. This feedback that she was incorrect was enough for her to return to the visual estimation strategy again on task 7 before using the length labels successfully on task 8. We suspect G3K1 wanted to utilize the length labels early, but the mixed feedback when utilizing the labels was difficult for her to interpret. Therefore, she primarily utilized visual estimation until she was certain that the length labels were productive. Furthermore, when she began utilizing the labels consistently we are not sure if she understood the correlation of volume units and length measurement or just the utility of the length label. Her explanation that the numbers give her, "clues," could show that she thinks of the numbers as being an arbitrary indicator of what numbers to use. She may not actually be coordinating a length measurement with a volume unit.

G3M1 stated she guessed on the paper portion of the early tasks but drew on the paper for task 1 and 4 in a way that suggested some visual measurement utilizing the cube. Her early predictions suggest that she was likely using the length labels in the DVM portion but used the wrong label to predict in two of the four tasks. After task 4, the interviewer asked if the length labels were helpful with the intent of hinting that the student should utilize the labels. Instead, we think this question may have had the opposite effect when coupled with the mixed results she received on her predictions. G3M1 began using her fingers as a measuring device on the subsequent tasks instead of attending to the length labels on both the paper and DVM portions of the tasks. She retained this strategy through task 9, despite understanding structure and interpretation as early as task 7. We think it may have been hard for her to release this strategy in favor of length labels because of the feedback she interpreted as pushing her to find a different strategy than using length labels. On task 9, she finally received adequate feedback to release this strategy in favor of using the length labels. She made predictions and built the entire prism according to her predictions before moving the built prism inside the prism frame. Although she previously received feedback that her predictions were incorrect, she had not seen the full result of her predictions. We believe that seeing the result of her predictions was enough to prompt a change in strategy, although feedback that predictions were incorrect was not.

4.7.2 Exceptionalities in the Interpretation Thread

In the interpretations thread, we found two types of persistent struggle. G3L2, who demonstrated the structure-representation pattern, had a persistent struggle with the interpretation thread, to the point that she did not reach the moment of change. We discuss her story further in section 4.7.3 because there is significant interaction among the structure and interpretations threads regarding her struggle. The other students who demonstrated a prolonged struggle with interpretation fit the (Structure-Representation)-Interpretation pattern. G3T2, G4N1, and G4C2 all persisted with surface area strategies on the paper portion of the tasks even after developing successful prediction strategies on the DVM portion of the tasks. The other three students who fit this pattern were relatively quick to transfer their understanding of structuring and appropriate use of the representation to the paper portion of the task, once they became successful in their prediction strategies. One of these students never used a surface area strategy, but the other two attempted to use partial surface area strategies in the early tasks. However, their surface area strategies were never as complete as those of G3T2, G4N1, and G4C2. To illustrate this challenge with surface area we share the story for G4N1.

On the first three tasks G4N1 used increasingly complete surface area strategies on the paper portion. She drew ten squares on the first task (see Figure M) and providing an apparently unrelated answer, but calculated the lateral surface area plus the base area correctly on tasks 2 and 3. For the DVM portion, she went from predicting the volume when seeing a layer on task 1, to predicting a row when seeing a cube on task 2, to predicting the entire sequence correctly when seeing a cube on task 3. She made all correct predictions for the DVM portion in every subsequent task except task 5, when we believe she switched the length and width labels. Thus, we identified her moment of change in the structure and representation threads on task 3. Her consistently correct predictions indicate that she was utilizing the length labels appropriately and conceptualizing the spatial structuring scheme modeled by the DVM.

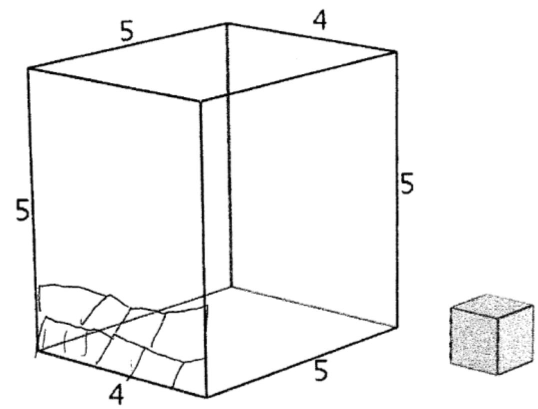


Figure M. *G4N1's Written Work on the Paper Portion of Task 1*

On task 4 G4N1 found the area of what she termed the front face and said the back would be the same. Then, she found the area of the base and, "multiplied that by 7 because I knew that would be going down all the way." This calculation was the correct volume, but she added the areas she found for the front and back faces to this value. This demonstrates that her spatial structuring scheme and understanding of the representation were present, but she was unable to coordinate them effectively because of her interpretation of the problem on the paper portion of the task. Thus, we interpret her continued utilization of surface area strategies as primarily an interpretive challenge rather than a structuring or representational challenge. However, we recognize that there may be interaction among the three threads causing some of the challenge.

On the next task, G4N1 again calculated the correct volume using the base area and height but added the area of all four lateral faces to that value. Before beginning the DVM portion of the task, the interviewer asked her if she expected the same result on the DVM and she did not. However, she did express that she wanted to get the same result. The interviewer also asked if she could tell what the volume would be on the DVM before asking the prediction questions. She did not answer correctly and made incorrect predictions, but only because she confused the length and width labels. After filling the prism, she was asked how many cubes were in the prism. She calculated a correct and complete surface area and was not able to resolve the difference in her paper calculation, the volume reported by the DVM, and her calculation of a complete surface area. We believe this sequence made her doubt the utility of the DVM portion of the task as a resource on the paper portion of each task and the validity of the volume reported by the DVM. Thus, she returned to her surface area focused strategy for the paper portion of subsequent tasks. However, she made correct predictions again on the remainder of the tasks.

On task 6, G4N1 calculated another partial surface area, but it is unclear which faces were included in her calculation. We note that she did not include the DVM's structuring scheme in her calculation on this task. After completing the DVM portion, the interviewer asked her about the actual volume and she used her own calculation instead of the value reported by the DVM, stating that the computer did not take into account the back side of the prism. The interviewer prompted her to explain how she predicted what the DVM would report even though she claimed the DVM did not report the correct volume. Instead of recounting her prediction strategy, she noted that on the paper portion she came up with the same volume as the DVM before she added one of the sides to her calculation. This is what made her think the DVM was not accounting for the back side. However, she changed her mind and decided she had added an extra side after thinking about which sides of the prism she included in her calculation. The interviewer did not ask any further questions because she was still utilizing surface area thinking to reason about the volume question.

On task 7, G4N1 used a complete surface area strategy. The interviewer prompted her to keep thinking about the fact that she predicted the volume and did not really need the DVM. On task 8 she claimed, "I'm gonna think about it as it's on the computer." She began by drawing three squares along a bottom edge (see Figure N), which would match with how the DVM creates a row. Then, she drew six squares along the adjacent bottom edge, which could match with how the DVM creates a layer. She counted the three squares, touching each once, then touched each of the six squares three times and claimed there would be 21 in the bottom. Although she counted an extra row, she appeared to be matching her mental structuring with the DVM's spatial structuring to this point. After this, she drew four squares up a vertical edge of the prism and six squares along the opposite edge of the same face of the prism where she drew the previous six squares. She counted her original six squares four times and stated there would be 24 on the back face. Then, she determined there would be 12 on two additional faces. Note that these face areas are correct, but she approached them with counting instead of the multiplication she previously employed. She added together the 21, 24, and two 12s to report 69 as the volume. Despite her purposeful intention to use a strategy based on the DVM structuring, she reverted to a partial surface area strategy after conceptualizing a layer. On the DVM portion, she made correct predictions throughout. After predicting the layer she corrected her thinking on the paper portion when she had predicted 21 in the bottom layer by counting an extra row. However, she used 18 instead of 21 in the same sum with the other face areas to correct her paper portion. She did not adjust the rest of her calculation to match the DVM's structuring of iterated layers.

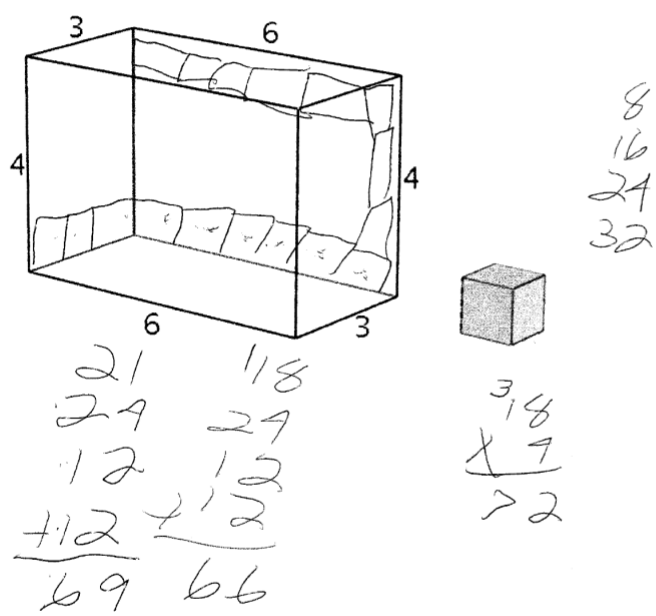


Figure N. *G4N1's Written Work for the Paper Portion of Task 8*

On task 9, G4N1 calculated the area of two opposite lateral faces of the prism as well as the base and top. She added those values together for the volume. Before asking about predictions, the interviewer asked her to describe what happens when using each button and use gestures to describe the DVM action. She predicted everything correctly on the DVM portion. When she had filled the bottom of the prism with a layer the interviewer commented that this wasn't the whole volume and asked, "What do we have so far?" G4N1 replied, "That's the bottom." Then, when she had iterated a layer only once the interviewer asked her what she had just done. She said, "I added a group of 30, because I had 30 at the bottom." Later the interviewer asked her more about her predictions on the DVM portion. He asked where the 30 had come from when she multiplied 30 times 3. She explained, "The 30 came from the 30 at the bottom." He asked, "But why multiply by 3 then?" G4N1 replied, "Because you have three going up. So, you have three groups of 30 going up."

We believe this conversation was instrumental in helping G4N1 reinterpret how to approach the question of volume. She was previously attempting to count how cubes fill space in two different ways on the paper and DVM portions of the tasks. She may not have recognized the difference in her counting schemes until verbalizing her approach. This exchange helped her recognize that the DVM showed how cubes could fill space to answer the volume question on the paper portion as well as on the DVM portion of each task.

On task 10, G4N1 produced a correct volume calculation, describing her use of, "rows of 15 going up" meaning layers. On task 11, she reverted to a surface area strategy initially, but changed her work for the paper portion to a correct volume calculation after beginning to make predictions on the DVM portion. Thus, we place her moment of change in the interpretation thread on task 10, when she first utilized the DVM's structuring strategy for the paper portion. We interpret her fallback on task 11 as a reversion to her prior interpretation that surface area structuring is useful for volume. She caught this reversion when she noticed the disconnect between her work on the paper and DVM portions because she had reinterpreted the questions as requiring the same solution strategy.

4.7.3 Exceptionalities in Structuring

In no cases did we identify students with what we would consider a prolonged struggle in the structuring thread. However, we recognize the interdependent relationship among threads, and identify two cases in which the structuring thread may have influenced a prolonged struggle in the interpretation thread. Both cases were mentioned in the interpretation thread.

In the first case, several students persisted with surface area strategies after demonstrating appropriate spatial structuring on the DVM portion of tasks early in the sequence. We note that these students held two spatial structuring schemes in tandem throughout many tasks. One scheme was modeled after the DVM's structuring using cubes. The other scheme included partial and sometimes full surface areas. Students consistently adjusted this scheme from task to task when it was ineffective in determining volume. Both schemes are useful spatial structuring schemes relating to three-dimensional space, but the students used the surface area scheme to determine volume on the paper portion of the tasks. We attribute this misapplication of the surface area structuring scheme to a misinterpretation of the question of volume. However, we also recognize that the continued use of surface area strategies may have been influenced by the structuring thread. Some students may have been able to mimic the DVM's actions to make numerical predictions without internalizing the spatial structuring scheme for mental actions on objects.

In the second case, G3L2 persisted with a spatial structuring scheme that was adequate to determine volume but did not match the DVM's spatial structuring scheme. Because she used vertical layers to determine the volume, she struggled to interpret the DVM portion of the tasks as analogous to the paper portion. In early tasks, she may have attempted to utilize her strategy from the paper portion on the DVM portion when she appeared to expect the DVM to match her strategy of creating a vertical layer. However, she abandoned her use of the paper portion on the DVM portion when she was unsuccessful at making accurate predictions that way. If she had been able to develop the appropriate associativity in her spatial structuring scheme to mentally create horizontal layers instead of vertical layers, we surmise that she would have made correct predictions on the DVM portion. Thus, the lack of associativity in her spatial structuring scheme may have created a prolonged struggle in the interpretation thread. Upon reflection, the interviewer should have allowed G3L2 to fill the prism on the DVM portion of at least one task by creating vertical layers. This may have helped her reinterpret the portions as analogous problems, solidify her spatial structuring scheme, and develop associativity in her spatial structuring scheme by demonstrating she was thinking about the problem differently than how the DVM was acting.

4.7.4 Exceptionalities in Enumeration

Although our analysis of the enumeration thread was limited, we found two informal results related to this thread. Both results are in relation to the computational fluency of the students. We found that the use of structure appeared to be delayed for multiple students who made frequent skip-counting errors or did not use multiplication efficiently, such as G3M1. We may attribute the appearance of delay to one of two reasons. The delay could have been empirically apparent because the student's enumeration errors masked the reality of the student's spatial structuring. The delay could have been a legitimate consequence of the interaction among the student's enumeration skills and mental spatial structuring schemes. We do not have strong enough evidence to support either conclusion decisively.

We also found that when students began using multiplication in place of skip counting, they appeared to dissociate the calculations from the structuring of composite units. Again, this appearance can be attributed to one of two reasons. The dissociation could have been empirically apparent because the strategy of multiplication did not require the demonstration of repeated units establishing a composite unit. Thus, the students may have still been conceptualizing the use of composite units without showing evidence of their thinking. Alternatively, the dissociation could have been a legitimate consequence of the interaction among the student's developing enumeration scheme and mental spatial structuring scheme. In this view, the spatial structuring became a latent element in the students' thinking that was no longer necessary when the use of multiplication developed. In order to reduce the cognitive load of the task the students no longer conceptualized the iteration of composite units, but instead reduced the task to computation alone.

5 conclusions

Volume calculation is a complex process because it requires the coordination of multiple understandings to produce accurate results. In this study, students had to be able to interpret the question of volume as one of counting cubes or volume units. They needed to be able to count efficiently and recognize the utility of repeated addition, skip-counting, or multiplication. They needed to understand the representation in which the problem is presented including the three-dimensional nature and the utility of length measurements for counting volume units. Finally, they needed to be able to coordinate these understandings through an efficient three-dimensional spatial structuring scheme.

We found the development of these understandings came in many different forms. We identified several patterns of development across the threads. We do not expect that these patterns are comprehensive. Additional students in other contexts may exhibit additional patterns of growth or alterations in these same patterns of growth. In this study the most prevalent pattern was (Structure-Representation)-Interpretation. However, we suspect that the conditions of the problem context and the treatment influenced the growth patterns so that in another context the most prevalent pattern might be different. For example, it is likely that the dynamic environment helped students develop spatial structuring skills on the DVM portion and then transfer those skills to the paper portion of the tasks. This would fit with this predominant pattern of growth. However, with the DVM portion of the tasks removed and replaced with a different treatment, it is unclear if a similar pattern of growth would occur.

We recognize that these four threads may not take precisely the same form in alternative volume measurement contexts such as the dynamic measurement of volume tasks proposed by Panorkou (2019). In these tasks, the researchers attempted to develop students' understanding of volume as the transformation of a two-dimensional area by pulling or dragging it through space. However, we believe these threads are likely still present even in this innovative approach to understanding volume. For example, students may not need to interpret the problem of volume as one of cubes filling space, because the focus is not on the use of cubic units. However, the students still need to interpret the question of volume in terms of filling three-dimensional space through the transformation of a two-dimensional area. Thus, the interpretation thread would still be necessary, but the moment of change would be identified differently.

We also found that the interactions among the interwoven threads of understandings were complex. First, in many cases students struggled to make progress in one thread because of the influence of another thread. Second, the moment of change in one thread often lead to or coincided with the moment of change in another thread. Third, aside from the influence of other threads, the type of feedback students received regarding each thread appeared to be influential in their developing understandings. Fourth, the spatial structuring thread proved to be unique in that students more persisted in with their spatial structuring schemes than with their understandings in other threads. Fifth, students' struggles with the traditional two-dimensional static representation of the paper portion had implications in several threads.

Many students struggled to make progress in one thread because of the influence of another thread. For example, G3L2 had developed an adequate, but inflexible understanding of spatial structuring by task 3 that may have influenced her prolonged struggle with interpretation throughout the remainder of the tasks. She struggled to coordinate the paper and DVM portions because her conceptualization of spatial structuring that was effective on the paper portion of the task did not match the model of structuring provided by the DVM. Because she was unable to develop flexible associativity in her structuring scheme, she was unable to coordinate the two problems as analogous. In other cases, students' lack of computational fluency in skip-counting and multiplication appeared to slow down their progress in the structuring thread.

In a similar vein, we found that a breakthrough in one thread often lead to a breakthrough in another thread. This is apparent in the patterns because only one student experienced the moment of change for each thread on separate tasks and in nearly every pattern at least two threads have a moment of change on the same task. For example, G3M1 demonstrated the moment of change in both the interpretation and structuring threads on task 7. In previous tasks she demonstrated that her spatial structuring was developing, but it was not solidified to the point that she could have produced a correct volume even if she utilized the length labels appropriately. On task 7, she demonstrated adequate spatial structuring on the paper portion of the task, although she did not produce a correct volume because she was not using the length labels. On the same task, she utilized her paper calculation on the DVM portion, demonstrating her moment of change in the interpretation thread. Although she may have been thinking that the two portions of the tasks were analogous in previous tasks, she lacked the confidence in her spatial structuring skills to act on this hunch. On task 7, she had solidified her spatial structuring skills in way that she was willing to utilize her results from the paper portion for the DVM portion. Thus, her moment of change in the structuring thread was the impetus for her moment of change in the interpretation thread.

We found that the type of feedback students received was often instrumental in their developing understandings. For example, when feedback was left solely to the volume report of the DVM, students often struggled to make progress in multiple threads simultaneously. The correct/incorrect feedback that students received from the DVM was adequate to make progress in a single thread. For example, students who quickly utilized the length labels and interpreted the questions as analogous soon developed spatial structuring without additional feedback from the interviewers. However, students who were working on multiple threads simultaneously often needed feedback from discussion with the interviewer to make progress. We suggest this is because students had a hard time identifying which part of their problem-solving process to adjust without additional feedback from the interviewer. We note that the influential feedback from the interviewers was typically questioning. Students were not told whether they were on the right track in any thread, instead the interviewers used appropriate questioning to help students identify positive and negative aspects of their reasoning. In most cases, such as the conversations with G4N1 on task 9, with G3C1 on task 4, G3M1 on task 4, and G3K1 on task 5 the questioning of the interviewer helped propel the students into better understanding. In a few cases, such as the conversation with G3C1 on task 1, the interviewer's questions may have caused a further delay in the student's development. Whether positive or negative this additional feedback was influential in the students' growth.

The spatial structuring thread stood out as unique because students tended to stick with a spatial structuring scheme even without seeing success. For example, G3K1, G3M1, G3L2, G4N1, G4T2, and G3C2 all utilized the same spatial structuring scheme for several tasks despite getting incorrect results. One of the reasons for this may have been the unique nature of the feedback on structuring. The DVM modeled an appropriate spatial structuring scheme, whereas with the other threads the only feedback from the DVM was whether the overall predictions were right or wrong. This may have allowed some students to solidify their understanding of the spatial structuring scheme and isolate other aspects of their problem-solving process to vary. For example, G3K1 developed an appropriate spatial structuring scheme that matched the DVM by task 4 but did not calculate the volume correctly or make correct predictions at that point. The DVM's feedback regarding structuring gave her confidence that her structuring was correct and allowed for to focus on changing other aspects of her problem solving until she was able to make correct predictions and calculations. G3M1 presents a similar case, in which the DVM feedback may have supported her correct structuring scheme. In the case of G3L2, she received positive feedback about her spatial structuring scheme because her volume calculations were correct. However, she was unable to predict on the DVM portion because her structuring scheme did not match the DVM's structuring. It is likely that the positive feedback about structuring was enough for her to retain her structuring scheme even though it did not appear effective in the DVM. G4N1, G4T2, and G3C2 all utilized surface area structuring schemes on the paper portion and volume structuring schemes on the DVM portion in tandem for several tasks. It is less clear why these students may have retained the surface area scheme for so long without any apparent positive feedback. Perhaps their prior experiences with area gave them confidence that their area schemes should be adequate to support their calculations on these tasks.

It is well-documented that students struggle to interpret traditional, static two-dimensional representations of three-dimensional objects ( ) and we found that the students' struggles with these representations had implications in several threads. The primary struggle here is understanding a two-dimensional representation of a three-dimensional object as viably three-dimensional. This is most obviously a struggle in the representation thread. Despite the difficulty in interpreting the traditional paper representation, we found the DVM helped bridge the gap of representational understanding, similar to ( ). However, our results also highlight that the dual nature of the dynamic representation does not automatically transfer to the static two-dimensional versions presented on paper. Students took time to recognize the representations as analogous and transfer their understanding of the dynamic environment to the paper version. Thus, dynamic environments are not a quick fix, but a tool to help students struggle through the coordination of multiple representations.

The struggle to transfer their understanding from the DVM representation to the paper represents a challenge in the interpretation of the question of volume. Our students needed to interpret the question as the number of cubes that would fill or build the prism. The two-dimensional representation did not directly encourage students to think three-dimensionally about cubes filling space. Instead, the DVM portion of the tasks and the initial connection with physical materials provided the clue that students should think about cubes filling space. Thus, the students needed to reinterpret the paper portion as analogous to the physical materials or the DVM portion in order to appropriately interpret the question of volume.

The struggle with representation also produced a struggle with structuring. Many students, including G3K1, G3M1, G4N1, G4T2, G3C2, and G4T1 utilized two dimensional forms of structuring prior to establishing appropriate three-dimensional spatial structuring schemes for volume. Drawings produced by G3K1 (Figures F to G) demonstrate her progression from strictly two-dimensional thinking to surface area based three-dimensional thinking and finally three-dimensional structuring for volume. G3M1's early drawings (Figure K) also represent strictly two-dimensional thinking, whereas G4T1's and G4N1's early drawings (Figures L, M, and N) represent surface-area based thinking. In all of these cases the two-dimensional representation influenced these students' spatial structuring schemes in the early tasks so they produced largely two-dimensional schemes with increasing understanding of three-dimensionality.

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Appendix A

Table A

Level Progressions in an LT for Volume by Student

	Task											
Student	1	2	3	4	5	6	7	8	9	10	11	12
G3K1												
Paper	NC	NC	NC	NC	VURR	VICS	VICS	VRCS	VRCS	VRCS	VRCS	VRCS
DVM	NC	VICS	VICS	VICS	VICS	VICS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS
G3C1												
Paper	NC	VURR	NC	NC	VICS	VRCS	VRCS	VRCS	VICS	VICS	VRCS	
DVM	VURR	VICS	VICS	VICS	VICS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	
G3T2												
Paper	NC	NC	NC	NC	NC	NC	NC	NC	NC	NC	NC	VRCS
DVM	VICS	VICS	VICS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS
G3M1												
Paper	NC	NC	NC	NC	VURR	VURR	VICS	VICS	VICS	VRCS	VRCS	VRCS
DVM	VICS	VURR	VURR	VURR	VICS	VICS	VICS	VICS	VICS	VRCS	VRCS	VRCS
G3L2												
Paper	VURR	VURR	VICS	VICS	NC	VRCS	VRCS	VRCS	VRCS	VICS	VICS	
DVM	NC	NC	NC	NC	VURR	VURR	VURR	VICS	VURR	VURR	VURR	
G3C3												
Paper	NC	VURR	VURR	VURR	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	
DVM	VICS	VICS	VICS	VICS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	
G3J3												
Paper	NC	NC	NC	NC	NC	NC	NC	NC	VRCS	VRCS	VRCS	
DVM	VURR	VICS	VICS	VICS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	
G4T4												
Paper	VURR	NC	VICS	NC	NC	NC	NC	VRCS	VRCS	VRCS	VRCS	
DVM	VURR	VICS	VICS	VICS	VICS	VRCS	VICS	VRCS	VRCS	VRCS	VRCS	
G4J1												
Paper	VICS	VICS	VICS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	
DVM	VICS	VICS	VICS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	

	Task											
Student	1	2	3	4	5	6	7	8	9	10	11	12
G4C4												
Paper	NC	VURR	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	
DVM	NC	VICS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	
G4T1												
Paper	NC	NC	VURR	NC	NC	VICS	VICS	VRCS	VRCS	VRCS	VRCS	VRCS
DVM	NC	VURR	VICS	VICS	VRCS	VICS	VICS	VRCS	VRCS	VRCS	VRCS	VRCS
G4N1												
Paper	NC	NC	NC	NC	NC	NC	NC	NC	NC	VRCS	NC	VRCS
DVM	VICS	VURR	VRCS	VRCS	VICS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS
G4C2												
Paper	NC	NC	NC	NC	NC	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	
DVM	VICS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	
G4L1												
Paper	NC	NC	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	
DVM	VICS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	
G4N2												
Paper	NC	NC	VRCS	VRCS	VRCS	VRCS	NC	VRCS	VRCS	VRCS	VRCS	VRCS
DVM	VICS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS	VRCS