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Learning Trajectories to Support the Development of Measurement Knowledge, pre-K through Middle School

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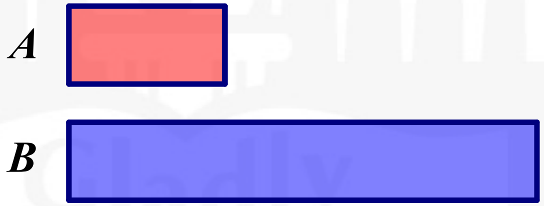
Measurement as Multiplicative Comparison

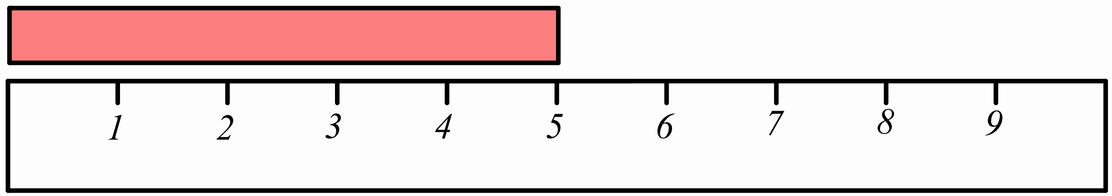
- Motivating number through continuous quantity and comparison



- B > A (Gross comparison)
- B is 2 more “As” (Additive)
- B = 3A or B/A=3 (Multiplicative)
  - Cite Piaget, Davydov – Barb, Slavin, Carraher,..

Interpreting All Measures as Comparisons?

- When a student reports a length of 5 inches, they need to see this as a multiplicative comparison between the object's length and 1 inch. "This strip is 5 times as long as a 1-inch strip."

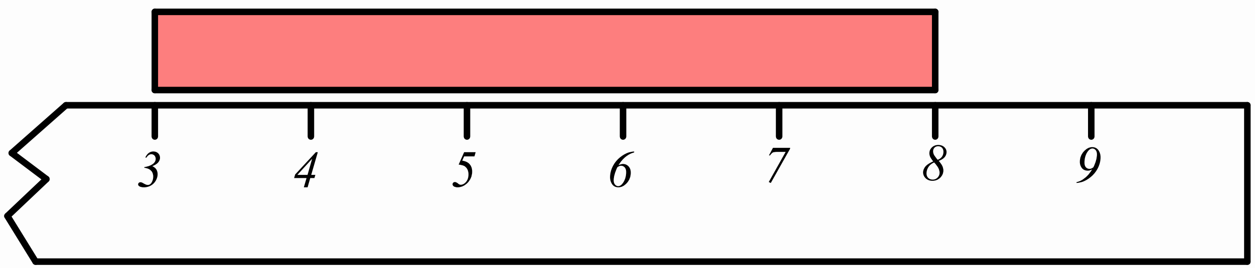


Interpreting All Measures as Comparisons?

- Missing the connection between measure and visual ratio?
  - Broken ruler
  - Triangle area
  - Number of units (area, volume) along an edge length
  - Area of circle (how many r^2 will cover?)

Missing the Connection

- Broken ruler: This strip is 8 inches long.
  - Does not interpret 8 inches long as being 8 times as long as 1 inch.



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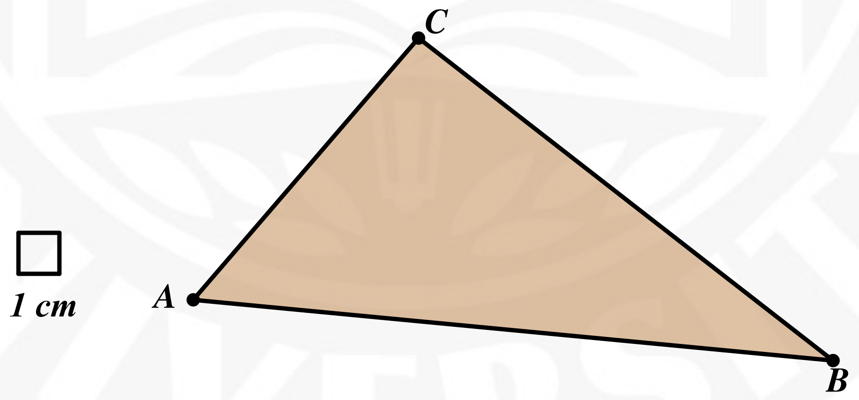
Missing the Connection

- Triangle area
  - Find the area of the triangle (54 square cm)
  - How many of these 1-cm squares would fit inside this triangle? (Maybe 32?)

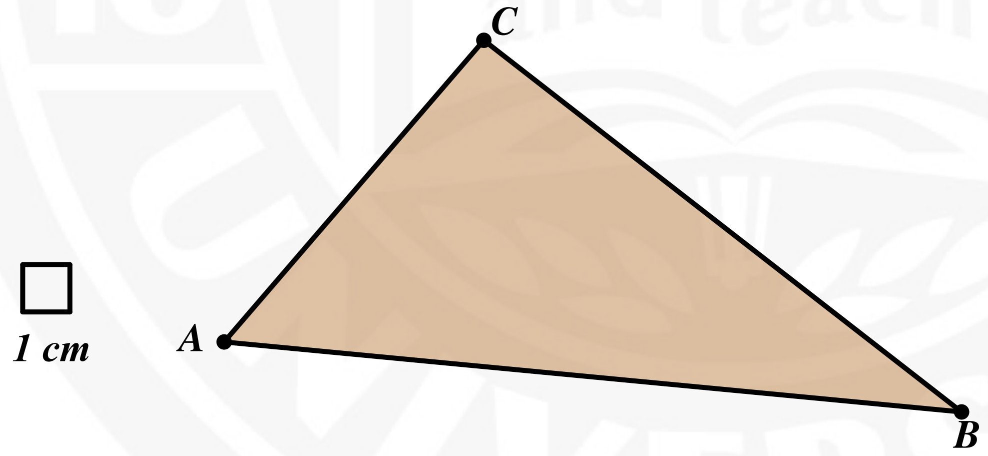
Area $\Delta ABC = 54 \text{ cm}^2$



Missing the Connection

- Number of units (area, volume) along an edge length
  - How many of the square cm will fit along this edge? (Maybe 10 or 11?)

$m\,\overline{AB} = 16$ cm



Missing the Connection

- Area of a circle
  - How do you find the area of a circle? ($A = \pi r^2$)
  - How many of these squares could fit inside this circle? (Maybe 3 or 4)

Bridging a Divide

- We see measurement as the intersection between symbolic operations on quantities and spatial comparisons between quantities.
  - We work to help students integrate comparisons (relationships) between quantities expressed symbolically AND spatially.

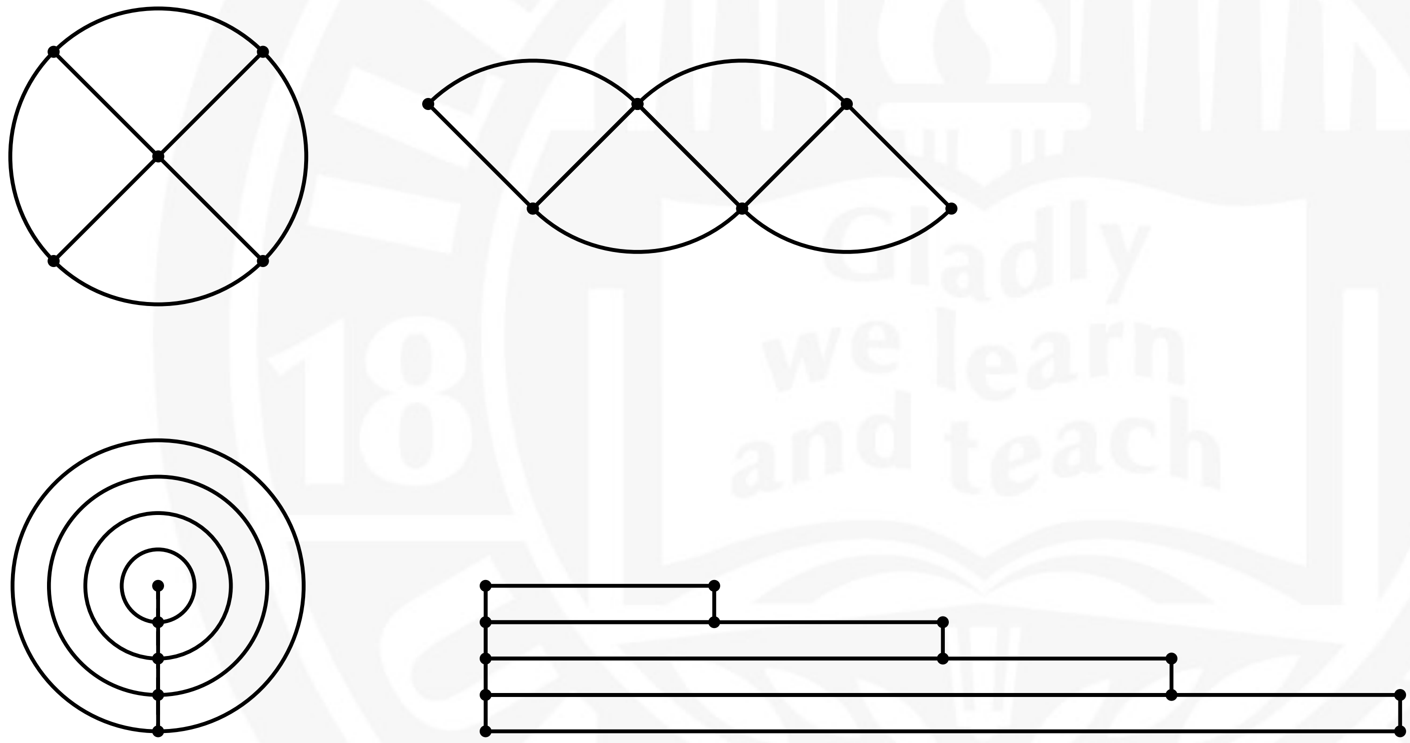
Understanding Area of Circles

- We wanted to examine students' ability to understand the formula $A = \pi r^2$ as a multiplicative comparison between the area of a circle and a square radius.

Understanding Area of Circles

- **Recall** the formula
- **Apply** the formula
- **Interpret** the formula
- **Defend/justify/prove** the formula

Circle Transformations



Affordances of Circle → Triangle

- Parallelogram decomposition
  - Circumference cut in half
  - Limit applies curved circumference to straight
  - Limit applies to angles turning into right angles

- Triangle decomposition
  - Circumference remains in one piece
  - Limit applies to "stair steps" to linear
  - The right angle between radius and circumference is immediate

Defend the measure of circle area: Identify the units of measure to structure the space

- Selected five sessions from our 3 year Teaching Experiment:
  - during Grade 8, November to February;
  - a cohort of four pairs of students.
- Data: Video records and written work from students with field notes.

Findings: recalled formula, stated quantity, yet without identifying units to fill the space

- They could recall and use the expression π r^2.
- Grasped circle area as a quantity of about 3 square units
- Studied a decomposition from concentric rings of a circle into incremental slats for a triangle:
  - base: circumference and altitude: radius

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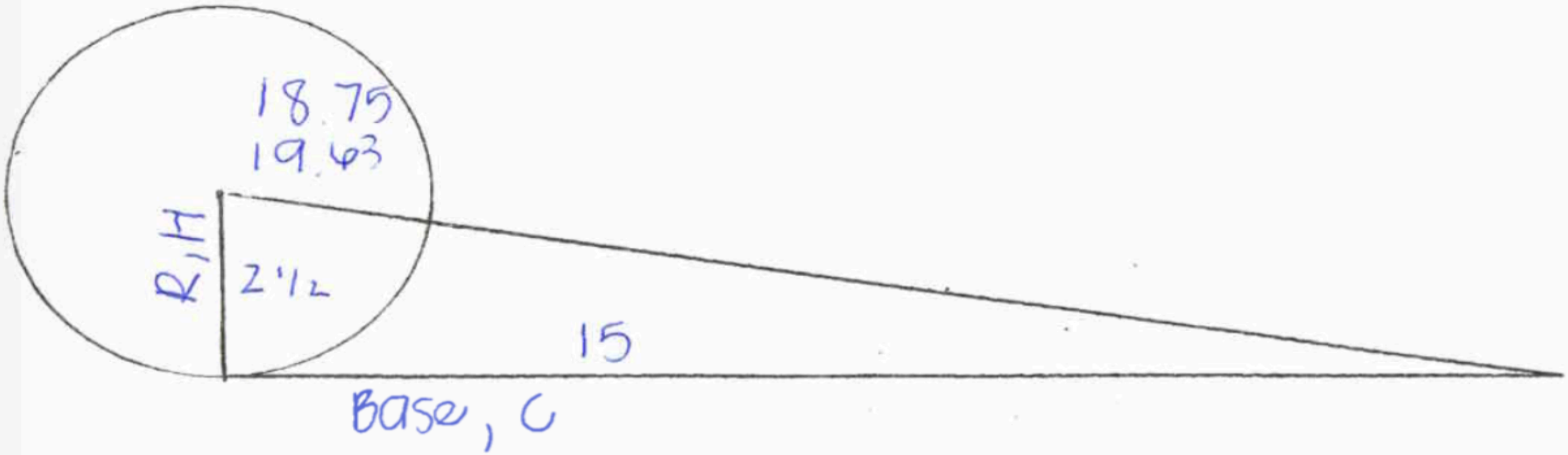
Relating radius to height, base to C



$rc \div 2 =$ area of triangle

$rc \cdot \frac{1}{2}$

Conclusions: triangle and circle mutually constituted

- The students said it made sense to them; but what did they conclude from it?

- The decomposition and re-composition was taken as **a valid** way to *relate* a circle to a triangle.

Successes: What did they get out of this?

- Circle → Triangle with same area
    - Area was conserved through a geometric transformation
  - Multiplicative relationship between circumference, radius and area ($A = c*r*1/2$)
  - $A = 2 \pi r * r * \frac{1}{2}$
  - How do these expressions relate?
    - Circumference = 2 π radius = triangle base
    - Radius of the circle = height of the triangle

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Struggles

- A=c*r*1/2 → A=pi*r^2
- They didn't see the beauty of deriving!
  - (Troubles with algebra – substitution and commuting to see 2*1/2 = 1)
- Confusion with the fact that "We already knew that!", to set the *direction* of an argument
- One of the four pairs of students did not like the idea of calling a triangle base "the circumference".

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Did they learn to justify the expression to find the circle area?

- They did not believe they had justified the standard expression for computing circle area.
- The direction of their argumentation was inconsistent with our expectation that they would state a rationale for the circle area formula.

Conclusions

Our findings suggest:

- The activity relating triangle and circle areas provided a supportive basis for linking measurement and multiplication operations.
- This example suggests the students forged a more comprehensive knowledge of both realms, the multiplicative reasoning, and their measurement sense for area.

Implications, questions:

- To what extent can students build a theory of measure for space before they have established multiplicative reasoning, geometric transformations, and algebraic reasoning?
- Measurement tasks in Grades 6-8 mathematics should engage students in finding the foundations and rationale for derived measures as a way of building their own theory of measure (e.g., the area of a circle, or the volume of a rectangular prism, or a parallelopiped)

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