LINKING CHILDREN'S KNOWLEDGE OF LENGTH MEASUREMENT TO THEIR USE OF DOUBLE NUMBER LINES

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In this paper we report on five Grade 6 students' responses to a proportional reasoning task. We conducted pair interviews within a longitudinal study focused on extending a hypothetical learning trajectory for length measurement. Results suggest that there exists a link between children's level of conceptual and procedural knowledge for length measurement and their ways of using the double number line representation when solving problems involving proportional reasoning.

INTRODUCTION

Researchers have recommended the use of double number lines in the teaching of various content domains (e.g., Kuchemann, Hodgen, & Brown, 2011; Orrill & Brown, 2012; Van den Heuvel-Panhuizen, 2003). In the United States, the Common Core State Standards (National Governors Association Center for Best Practices, & Council of Chief State School Officers, 2010) specifically recommends using double number lines in the teaching and learning of ratio and proportional reasoning.

Van den Heuvel-Panhuizen (2003) explored the didactical use of a form of a double number line, the bar model. In her work she found that this form of a double number line "can function on different levels of understanding, and that it can keep pace with the long-term learning process that students have to pass through" (p. 30). Kuchemann, Hodgen, and Brown (2011) argued that an understanding of the double number line model is important for helping students make a shift in understanding multiplication as scaling. They also noted that, much of the work relating to the double number line model has been focused on its use as a support for teaching.

In their work, Orrill and Brown (2012) identified conceptual foundations, coordinating units and partitioning, as critical pieces of knowledge for using the double number line representation to support proportional reasoning. Aside from this work, little is known about what concepts and processes are needed to develop fluency with the double number line model. The purpose of this report is to address this gap in the literature.

RESEARCH QUESTION

How does children's knowledge of measurement relate to their ability to use double number lines when solving problems involving proportional reasoning?

THEORETICAL FRAMEWORK

The purpose of this study was to explore children's knowledge of length and how it relates to their use of double number lines while solving proportional reasoning problems. Thus, we needed a theoretical tool that allowed us to describe and differentiate children's knowledge. A hypothetical learning trajectory (LT) for length measurement served this purpose. An LT has three parts: (a) an instructional goal, (b) a likely path for learning, and (c) the instructional tasks that support children's growth through those levels (Clements et al., accepted under review).

LTs are a central feature of hierarchic interactionalism (HI), a theory of cognitive development that integrates empiricism, (neo)nativism, and interactionalism (Clements et al., accepted under review). LTs originate from HI, which postulates that children progress through domain-specific levels in ways that can be characterized by specific mental objects and actions (i.e., both concept and process) that build hierarchically on previous levels (Clements et al., accepted under review).

The following length LT levels (Clements et al., accepted under review) are relevant to the present study.

Length Unit Relater and Repeater (LURR): Children at this level measure by repeating, or iterating, a unit, and understand the relationship between the size and number of units.

Consistent length Measurer (CLM): Children at this level see length as a ratio comparison between a unit and an object. They use equal-length units, understand the zero point on the ruler, and can partition units to make use of units and subunits.

Conceptual Ruler Measurer (CR): Children develop schemes for mentally iterating, partitioning, and unitizing in tandem with a coordinating space and number scheme.

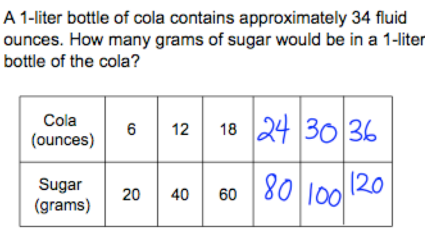
Integrated Conceptual Path Measurer (ICPM): Children incorporate multiple units and collections of units and operate on sub- and super-ordinate units. They have the ability to compensate within a single scale; however, they do not coordinate a series of changes in a systematic way across multiple scales to formulate and justify a valid argument.

Coordinated, Integrated Abstract Measurer with Derived Units (CIAM): At this level, children coordinate multiplicative and additive reasoning in fluent ways and engage in proportional reasoning about repeated or coordinated cases. In addition, they are able to reflect on derived units as an attribute.

METHODOLOGY

The design of the present study[1] was informed by previous work for extending LTs for measurement (Clements et al., accepted under review; Kara, 2013). This organizing methodological structure includes a) posing tasks that reveal children's thinking about a concept outside the LT, b) presenting the tasks to children in the same and adjacent LT levels, c) differentiating children's responses, and d) comparing strategies of children within and across levels to inform extensions to the existing LT.

We focused on five sixth grade children from a public school in the USA. Data were collected over a two-month period as part of a longitudinal teaching experiment (Steffe & Thompson, 2000). We collected data midway through a four-week unit focused on ratios and proportional reasoning. In the two class periods preceding the data collection, instruction focused on transitioning from using tables of values to double number lines. The following illustrates the teacher's instructional sequence of transitioning from a table of values (Figure 1) to a double number line (Figure 2) and zooming in to find a target value on a double number line (Figure 3).


A 1-liter bottle of cola contains approximately 34 fluid ounces. How many grams of sugar would be in a 1-liter bottle of the cola?

Cola (ounces)	6	12	18	24	30	36
Sugar (grams)	20	40	60	80	100	120

Figure 1: Table Representation

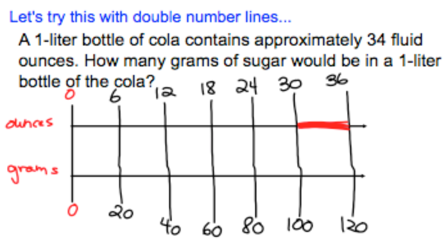


Figure 2: Transition to Double Number Line

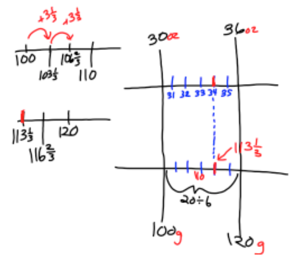


Figure 3: Zooming in to Find a Target Value

The data sources for this report included three 30-minute semi-structured pair interviews and one written assessment. We coded the assessments by LT levels and generated predictions based on these codes. The interviews were videotaped and

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transcribed. We compared children's responses from the interviews to the predictions to map double number line strategies into the LT.

RESULTS AND DISCUSSION

Predictions Based on the Written Assessment

Based on the written assessment, we identified students at the levels LURR and CLM. On these initial assessment items, two students (Chris and Martha) exhibited LURR level thinking, and three students (Mia, Karen, and Carrie) showed they were operating at least at the CLM level of the length LT. During the set of precursory interviews, Carrie often made use of LURR level strategies; therefore, the research team determined that she was predominately operating at the LURR level. Similar interactions with Chris, Martha, Mia, and Karen provided further evidence that their level placements based on their initial assessments were accurate.

In our prior work, we saw LURR and CLM level thinking predominantly in Grades 2 and 3 (Clements et al., accepted under review). We hypothesize that the Grade 6 students in the present study exhibited LURR and CLM level thinking because the tasks required students to integrate number knowledge and measurement knowledge with ratio reasoning. We think this introduced a level of complexity to the task that might have prompted students to revert back to lower level strategies (Sigler, 1986).

Different LT levels are characterized by specific mental objects and actions (Clements et al., accepted under review); therefore, our research team predicted that students at adjacent levels would use double number lines in different ways. According to the length LT (Clements et al., accepted under review) students at the LURR level measure by repeating, or iterating, a unit; therefore, we expected students at this level to rely on an iterative strategy. Students that are at least the CLM level see a measurement as a ratio between a unit and a length to measure, and can partition units to make use of units and subunits. Hence, we expected to see students who are at least at the CLM level correctly attend to units along one scale, and apply a partitioning strategy. Furthermore, we looked for evidence that they could coordinate units along two scales simultaneously as evidence of concepts and processes of higher LT levels (ICPM or CIAM).

At the beginning of the interview, each student was given the following problem printed on a worksheet:

> While shopping, Kyla found a dress that she would really like, but it costs $52.25 more than she has. Kyla charges $5.50 an hour for babysitting. She wants to figure out how many hours she must babysit to earn $52.25 to buy the dress. Use a double number line to support your answer.
>
> (http://commoncore.org/maps/images/math_documents/G6-M1-Student_Materials_(Eureka_Version).pdf)

The following sections present pairs of students' responses to this task.

LURR Level Pair

Carrie and Martha initially created a table of values, ranging from 1 to 5 for hours and $5.50 to $27.50 for dollars earned (see Figure 4). This suggests that both Carrie and Martha could correctly apply the unit rate of $5.50 per 1 hour to create a table by iteration of units.

Carrie then asked, “Where are we going to?” They settled on a target value of $26.00 as Carrie explained “she wants to buy a dress that’s fifty-two dollars and twenty-five cents, so we figured half of fifty-two is twenty-six dollars and so we’d have to find someplace in between twenty-two dollars and twenty-seven is twenty-six and then when we find our answer, then we’ll just double our answer because that’s half of fifty-two.” Carrie and Martha then both drew a double number line, labeling one line as hours and the other as dollars earned. At this point in their solution process, Carrie and Martha were attending to units along only one scale, dollars earned.

When asked what they would do next, Martha explained that they usually make markings in between the tick marks. Carrie said, “Since no numbers are between four and five, we can’t put any markings up here (pointing to the hours line).” Martha then said, “so, we’ll do this one” (pointing to the dollars line) and told Carrie that they needed to find a number that “goes equally” in the interval between $22.00 and $27.50. Because of Carrie and Martha’s discussion of both number lines, the interviewer suspected a transition in their thinking from attending to units along only one scale to coordinating units along two scales simultaneously. Therefore, the interviewer asked how many hours Kyla would need to work to earn the total amount needed for the dress, so the students returned to their tables and extended them as shown in Figure 4.

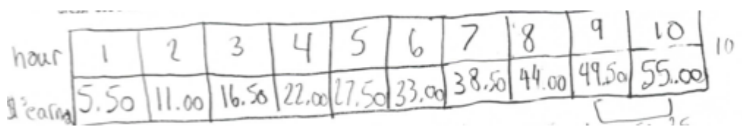

Figure 4: Carrie’s Table of Values

The interviewer then asked about the location of $52.25. Carrie explained it was between $55.00 and $49.50. She extended her double number line and created two tick marks on the hours line, and labeled them 9 and 10. Next, Carrie made corresponding tick marks on the dollars line, and labeled them $49.50 and $55.00. Carrie said she would have to make tick marks between these two values. Next, Carrie and Martha applied an iterative strategy. They tried counting by various dollar amounts ($1.00, $1.50, $1.25, $0.50, and finally $0.75). Each time they rejected the value because they could not reach their target value and the $55.00 tick mark. Due to time constraints, the interview ended before Martha and Carrie reached a solution.

LURR and CLM Level Pair

Chris and Karen began solving the problem by creating a table. Using this representation, they were able to correctly apply the unit rate of $5.50 per 1 hour to create a table. When Karen had extended her table beyond 5 hours, she was asked whether she needed to go by one hour or if she could put a 10 in the next box. She explained that she could go from 1 to 5 hours and then double the value for the dollars earned for working 5 hours to get the value for 10 hours. She then subtracted $5.50 to determine the dollar amount that would correspond to 9 hours.

Next, Karen and Chris created a double number line representation to zoom in on a target value (see Figures 5 and 6). Karen then applied a partitioning strategy to this region of the double number line as she drew a tick mark between her tick marks labeled as 9 and 10 on the hours line and connected it to a tick mark on the dollars line. This suggests that, as Karen applied this partitioning strategy, she was able to coordinate units along two scales simultaneously.

Karen then said, “If she worked for 9 hours and 30 minutes, how much will she get?” She labeled the tick mark on the hours line as 9 hours and 30 minutes and recalled that each interval on the dollars line represented $5.50. With computational help from the interviewer, she divided $5.50 by 2 to get $2.75. Next, she asked the interviewer how she could find out the value of the tick mark on the dollars line that corresponded to the tick mark labeled as 9 hours and 30 minutes on the hours line. The interviewer told her it meant that she needed to go $2.75 more than the dollar amount that corresponded to 9 hours, and she added $2.75 to the $49.50 and got $52.25, which she realized was her target value.

Chris followed Karen’s partitioning strategy. However, he did not immediately recognize that he had reached the target value, and he continued partitioning the two regions to the left and right of the tick mark labeled as 9 hours and 30 minutes on the hours number line. This suggests that Chris, who had been placed at the LURR level was not able to maintain the coordination of units along two scales simultaneously when applying the partitioning strategy.

CLM Pair

As Mia initially engaged in the task, she drew a double number line and created tick marks on the dollars line with intervals of $5.50 and tick marks on the hours line with intervals of 1 hour. However, she did not maintain even spacing as she drew tick marks along both number lines. This became problematic for her, when she applied the zoom in strategy. She drew a second zoomed in number line, with tick marks labeled as $49.50 and $55.00 on the dollars line. At this point, Mia paused. To prompt her to think about labeling the corresponding tick marks on the hours line, the interviewer asked, “What matches on the bottom of your other number line?” Mia returned to her original number line and labeled more of the tick marks on the dollars line. She then said, “I got a 7” (pointing to the tick mark on the hours line

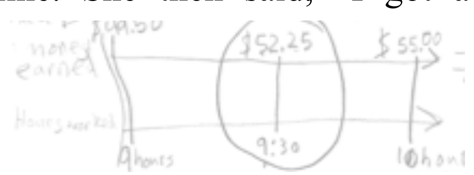

Figure 5: Karen’s Partitioning Strategy

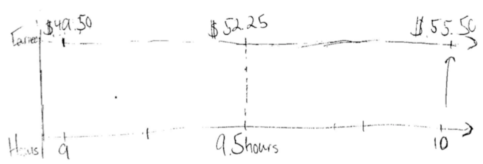

Figure 6: Chris’ Partitioning Strategy

corresponded to the tick mark labeled as $49.50 on the dollars line). Therefore, Mia showed that she could attend to units along one scale.

To help her shift to thinking about coordinating units along two scales simultaneously, the interviewer suggested that Mia draw segments connecting each labeled tick mark on the dollars line to a labeled tick mark on the hours line on her original double number line. The interviewer again asked how many hours corresponded to the value of $49.50. Mia then indicated on her zoomed in number line that the $49.50 tick mark corresponds 9 hours, and the $55.00 tick mark corresponds 10 hours. Next, Mia set out to “find in between of $49.50 and $55.00.”

The interviewer then suggested that she show where her target value of $52.25 would be, but Mia said, “I don’t know.” When asked how much more $55.00 was than $49.50 Mia said, “Five and a half.” Next, the interviewer suggested they break this piece of the number line into pieces. Mia initially suggested that they create five pieces. Mia’s partner then drew in five tick marks (and later corrected to four) between the tick marks labeled as nine and 10 hours. Mia and her partner labeled the tick marks as nine and one fifth to nine and four fifths. They assigned a value of one fifth of an hour to each interval they created on the hours line; however, they did not apply a partitioning into fifths on the dollars number line. Instead, they reverted back to an iterative strategy, trying to pick a unit that would allow them to span from $49.50 to $55.00 In other words, Mia and her partner could track units (1/5 of an hour) along one scale, but they did not coordinate units along two scales simultaneously. We are not sure if this is because dividing $5.50 is difficult or because they were unable to coordinate.

Mia’s partner suggested splitting the interval in half. At first, Mia said she could not split the interval in half because there were five “things.” However, when the interviewer asked how much was in the interval from $49.50 to $55.00, Mia said “five and a half.” When the interviewer again asked if she could split it in half, Mia said “yeah,” stating it would be $2.75. Mia explained that the $2.75 represents the halfway point between $49.50 and $55.00. Mia added $2.75 to $49.50 to get $52.25. The interviewer then asked how many hours it would be, and Mia correctly said nine and a half hours. Mia was able to coordinate units along two scales simultaneously with support from the interviewer and only when operating on halves.

CONCLUSIONS AND IMPLICATIONS

Findings suggest a link between length LT level and children’s use of double number lines when solving proportional reasoning tasks. The LURR pair, Carrie and Martha, predominantly relied on an iterative strategy, which is consistent with our prediction. That is, they applied a unit rate by iteration of units to the table representation, and an iterative strategy, of counting by various dollar amounts, to the double line. They also exhibited a lack of understanding of the density of the number line when they noted that there were no numbers between four and five. We conjecture that this is why they did not partition the double number line, which is a CLM level strategy.

The CLM pair, which included Mia, was able to attend to units along one scale and apply a partitioning strategy. However, they could not coordinate units along two scales simultaneously without the interviewer's expert scaffolding. Chris, who was part of the LURR and CLM pair, followed along with his CLM-level partner's (Karen's) partitioning strategy. However, his willingness to continue partitioning the hours line, without checking to see that he had reached the target value on the dollar line, suggests that he was unable to coordinate units along two scales simultaneously. Mia and Chris' strategies were consistent with our prediction for students at the CLM level of the length LT.

Although not initially placed at the CIAM level, Karen exhibited concepts and processes consistent with this level as she engaged with the double number line representation. For example, she applied a partitioning strategy while maintaining the coordination of units along two scales simultaneously without prompting or support from the interviewer. We take this as evidence that Karen may be operating higher than the CLM level of the length LT. In particular, we think Karen's simultaneous coordination of units along two scales exemplifies proportional reasoning about repeated or coordinated cases, which is consistent with the CIAM level of the length LT. Although we did not see her exhibit a reflection on a derived unit as an attribute, we conjecture that the task did not require this reflection.

Parallel to prior research, this study established the importance of an understanding partitioning and coordinating units (Orrill & Brown, 2012) for understanding the double number line representation. However, in the present study we established a link between the levels of an LT for length measurement and students' ability to use the double number line representation when solving proportional reasoning tasks. In particular, our prediction that students at the LURR level would rely on iterative strategies, and children at the CLM level would partition and correctly attend to units along one scale, but not yet coordinate units along two scales simultaneously were correct. Future research is needed to explore ways to support children at LURR and CLM levels in developing these concepts and processes.

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