USING DESIGN-BASED TASKS TO TEACH AREA MEASUREMENT

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Geometric measurement is a critical domain that is difficult for many students. The focus of this study was to determine if the incorporation of design processes into instructional activities for area measurement may enhance engagement and learning of students from low-resource, historically marginalized communities. We adapted activities from a learning trajectory for area measurement, prompting Grade 3 students to integrate knowledge of arrays, multiplication, and area measurement. Results suggest the design focus prompted students' integration of knowledge of space and number by engaging in novel representations of designed objects such as parking lots. Design-based tasks on measurement also prompted multiplicative thinking among students who had been less engaged in tasks involving arrays to represent multiplication.

Keywords: Design Experiment, Elementary School Education, Geometry and Geometrical and Spatial Thinking, Learning Trajectories

We examined the use of design and measurement topics as a means of reaching into complex cultural contexts with elementary age students of mathematics. Alan Bishop (1988) considered mathematics a cultural product. He noted that people in most cultures engage in six fundamental mathematical activities as they develop mathematical knowledge: counting, locating, measuring, designing, playing and explaining. Similarly, "[a]ll science learning can be understood as a cultural accomplishment" (National Research Council [NRC], 2012, p. 283). People explore their world using various activities related to science and engineering (NRC, 2012). The Common Core Standards for Mathematics recommends high schools students use geometry to solve design problems (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010). However, the Next Generation of Science Standards (NGSS Lead States, 2013) consider constructing explanations and designing solutions a vital practice for all students to learn to prepare for STEM fields. At Grades 3 – 5, the standards recommend students "Generate and compare multiple possible solutions to a problem based on how well each is likely to meet the criteria and constraints of the problem" (NGSS Lead States, 2013, p.46). Thus, we considered design problems an accessible instructional consideration suitable for elementary students.

We selected area measurement and design problems to make use of two of the six fundamental activities identified by Bishop (1988) in an attempt to enhance existing mathematics instruction for elementary students. We relied on research that produced a learning trajectory (LT) for area measurement (Barrett, Clements, & Sarama, 2017) for a theoretically and empirically validated cognitive account of plausible assessments and instruction (Barrett & Battista, 2014; Battista, 2006, 2012; Sarama & Clements, 2009). Instructional task sequences from these LTs offer materials to support teachers in helping students achieve their mathematical potential. However, there is a need to investigate how developed materials can be adapted to better serve under-supported groups of students. Research programs somewhat similar to those producing the LTs, such as Cognitively-Guided Instruction (Carpenter & Fennema, 1992; Fennema, Carpenter, &

Franke, 1997) benefited from productive adaptations (Brown & Campione, 1996) as they built on students' oft-neglected community funds of knowledge (Celedòn-Pattichis et al., 2018; Wager & Carpenter, 2012).

Thus, we had two related aims for this study: (a) determine if adaptations of instructional activities based on an LT for area measurement (i.e., Barrett, Cullen, Behnke & Klanderman, 2017, pp. 127-137) enhance the engagement and learning of underrepresented students and (b) find whether design processes enrich established instructional interventions. To provide appropriate instruction, we sought ways to engage the students in design-based tasks to measure large 2D spaces. We expected such tasks to help students anticipate collections of units as effective tools for measuring space, through multiplication or addition. By anticipating the use of collections of units, students might begin to rely on skip counting, and transition toward multiplicative reasoning in an array structure. We sought to emphasize the benefits of arrays as models for counting units of area. We expect to suggest a model for improving the development of asset-based LTs that bridge cultural, community-based practices among elementary students. This was our rationale for adapting existing LT instructional tasks to (1) feature design processes, and, (2) integrate multiplication operations, arrays and area measurement.

Method

Participants were a convenience sample of twenty-two Grade 3 students in an urban Midwest classroom and their teacher. Their school district consists of approximately 13,000 students (20.1% White, 57.7% Black, 11.3% Hispanic). Approximately 68% of the students at the school receive free or reduced lunch and 8% are English learners. Based on a brief written assessment using published area measurement tasks (Barrett, Clements, & Sarama, 2017, pp. 105-115), classroom observations, and instructional plans from a published LT (Barrett et al. 2017), we found four levels of thinking among these 22 students. Three students exhibited Physical Coverer and Counter strategy, meaning they may cover a rectangular region given enough physical tiles, but struggle to count without actual tiles. Five showed Complete Coverer and Counter (CCC) strategies, as they covered a rectangular region without leaving gaps or overlapping and placed unit shapes in lines. Nine students demonstrated Area Unit Relater and Repeater (AURR), showing an intuitive sense of rows and columns when drawing units to cover a region; they used consistently sized units. Finally, three students used Initial Composite Structurer strategies. They recognized a square unit and began to organized units into coordinated collections of units they treated as a group. We decided to focus instructional work to fit two LT levels: Complete Counter and Coverer level and the Area Unit Repeater and Relater level. We expected many of the students to engage units of area measurement to cover a region, either by physically placing units together to fill, or by drawing units to span a region. The unit organization of students exhibiting these levels is typically disjointed. At their best, they would be lining up several objects in loosely related stacks to attempt to cover the space being measured. Once covered, students functioning at these levels often count the squares to measure, perhaps seeing them as "square units" (at the AURR level). We set out to engage students at such levels on the area measurement LT in design work, expecting to help them see the value of arrays and multiplication strategies for measuring area.

Instruction Design Cycle

Prior to instruction, the researchers observed and worked with students in the classroom to build familiarity and rapport during three visits to the classroom. Later, we interviewed students in focus groups. We asked them to tell us how they may already use mathematics outside of the classroom to count, locate, measure, design, play or explain (Bishop, 1988). Our instruction design cycle consisted of a series of three lessons administered during one week of school in the Fall of 2019. Each lesson was led by one of the authors, with assistance from the classroom teacher. The first lesson was an adaptation of patio tasks targeting LT levels often found among Grade 3 students (Barrett et al., 2017, p. 133-137). The two later lessons involved a process of reflection on the difference between a designed task and our observations from the implementation of the tasks from lesson 1, leading to lesson 2 and finally lesson 3.

What we report here is the feasibility study phase of a design experiment (Middleton, Gorard, Taylor & Bannan-Ritland, 2008). This phase is meant to evaluate an intervention through qualitative methods such as observations, interviews, and case studies to determine what aspects of the intervention work and those that need improvement. Next, we must decide how best to proceed to a formal teaching experiment.

Each lesson began with whole-class discussion of a complex measurement question on area. First, we set a designing task using a novel problem, to find the number of buses that could fit in a parking lot (day 1). Next, we asked them to design and draw a parking lot to fit some given number of cars or buses (day 2). Finally, we ask students to design and draw a park for pets, to provide room for some given number of dogs to move around freely for exercise (day 3). The teacher and researchers surveyed students' progress by assisting students who asked questions and posing questions to students while they worked. Students worked independently at first, and later in teams of two or four. The researchers kept field notes. Student work was collected for analysis. At the end of each lesson, the researchers reflected on what occurred in the class to develop the goal and a focal task for the subsequent lesson.

For the first day we initially showed the students an aerial picture of their school. Students immediately identified the image as their school and identified certain areas of the school including the playground, sandlots and parking lot. Next, we presented students with an image of the school parking lot along with a rectangle that represented an approximate scale version of a school bus. We asked the students to determine how many buses would fill the school parking lot. After students completed the task working independently, we had some students present their solutions. We ended the lesson with a discussion of how to count the parking spaces in the lot without counting all spaces. After one student presented his approach to grouping collections of parking spots in a row, the instructor emphasized his gesture and asked him to recap how he counted by groups, with each successive sweeping motion along rows of parking spots.

In reflecting on the task presented and preparing for the second lesson, we decided the process of filling the parking lot was made more challenging since the "bus" was an immovable image (recall that students were shown a rectangle image to represent the size of one bus, drawn outside the parking lot area). Some students had used pieces of paper to represent the bus in order to show multiple positions and spaces for a bus in that first task. We also wanted to build on an idea that one student presented to the class during the first lesson. He used his hands to motion along a row of buses at the top of the board in front of the classroom. As he motioned thus, he said would be able to count one “whole row” of parking spaces as 20, and then add the same number again for each of the other rows. He said there would be 3 rows of 20. We felt that grouping sets of buses, taken in rows, to fill up a parking lot would be a good step towards creating arrays. We also expected this kind of reasoning to support a link with students’ multiplication operation, or at least link to their skip-counting operations for addition of same-size groups.

The second day, we reviewed the ideas discussed in the first lesson. The instructor reviewed the idea of the sweeping motion as a strategy for counting in a large set of objects like cars in a parking lot, using rows as a way to “skip count” by the number of cars in that row. Then he showed students images of several different parking lots and asked them how many spaces were available for cars to use. He asked, “Can you find out without counting each space individually?” Next, we gave students a rectangular cut out of a bus (1 x 4 cm) and car (1 x 2 cm). We asked them to use an 8 ½ x 11-inch sheet of paper to draw their design for a lot. Students then were given the task of creating a parking lot for cars [or buses] with the condition that the lot will hold 25 [or 40 or 60] vehicles. Several students presented their designs at the end of the lesson.

In reflecting on the second lesson, we saw that the cut-out vehicles helped many students in the design of their parking lot. We also realized that a parking lot context may have limited the ability to consider various arrays because typically parking spaces at best are 2 x n arrays. We felt we needed to change the design task so that various area arrays were possible. We decided designing a dog park would be the context we would use for a third lesson.

The third day of instruction, we presented students with an image of a dog kennel. In the image it showed that a 6 x 6 foot square was an adequate area for a dog to run around in. We mapped out the square area on the floor of the classroom so students could see the space, walking around inside the mapped-out region to show the space needed per dog. We gave students a 1.5 x1.5 inch square cutout piece of cardboard. We told the students it represented the space that one dog needed to move around. The task we posed for them was to design a rectangular dog park that had enough room for 24 dogs. At the end of the lesson, we had students present their designs.

For this third lesson, we realized that the context was problematic as it related weakly to the mathematics. We were interested in providing a context in which various arrays would be possible. The area needed for a dog to run seemed promising to us. However, dogs do not stay in one location. They end up sharing an entire area, and still they must be accounted for individually. So, the idea of a square area as a tool representing the space needed for a dog apparently did not make sense to several students. Some students made larger parks; we believe they did this to give the dogs a maximum area for playing.

Table 1: Strategies for Filling Parking Lot

Strategies	Number of Students (n=22)	Estimated Number of Buses
Partially Filled Lot with Fairly Consistent Units	7	62 to 100
Filled Lot Using Existing Lot Lines	6	60 to 92
Filled Lot but with Inconsistent Units	6	10 to 80
No Strategies Stated or Demonstrated	3	2 to 9

We analyzed the students' work in two ways. First, we examined all three tasks to determine strategies used to solve each. We asked ourselves how students made use of arrays or units in developing solutions. For the second and third tasks, we examined whether the students met the constraints of the task in the process of designing a solution.

Results

Table 1 (above) summarizes the strategies students used to determine how many buses would fit into the space of the school parking lot. Most (n = 19) of the students demonstrated a strategy for determining the number of buses that filled the parking lot. The other students (n = 3) did not demonstrate a strategy or state a reason for the number of buses they claimed would fit in the lot. Of those with evident strategies, 13 students made use of fairly consistent units by either using the parking lot spaces to count or drawing their own bus units to cover the lot. The remaining students used inconsistent units to fill the lot. Figure 1 shows drawings representing three strategies we identified.

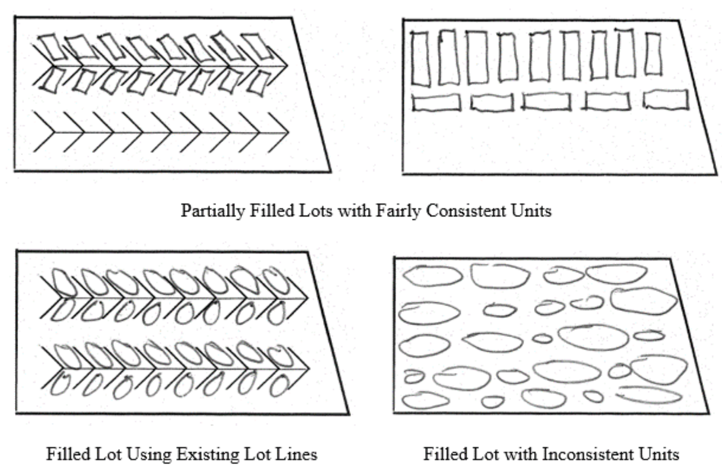

Figure 1. Strategies Used to Fill Parking Lots (from day 1)

Table 2 summarizes the strategies used by students to create a parking lot on day 2, based on bus-sized or car-sized units, and the number of students in each group that met the constraint of designing a parking lot with a reasonable number of spaces of the given size for a "unit". We believe that more than half the students made use of grouping and arrays to create their designs. Almost all of the students (n = 20) used a consistent size and shape to build up their parking lot designs. Figure 2 shows examples of five prominent strategies for checking the number of buses (or cars) the parking lot would hold.

Table 2: Strategies Used in Designing Parking Lot

Strategies	Number of Students (n=22)	Lot Size Constraint
Consistent Units with Some Use of Arrays	5	4
Consistent Units with Grouping	7	3
Consistent Units with No Grouping	6	5
Consistent Units with 1 Row of Units	2	0
Inconsistent Units No Clear Spatial Arrangement	2	2

Fourteen students met the lot size constraint. No particular strategy use resulted in all students in that group meeting the lot size constraint (25, 40 or 60). It should be noted that we considered total spaces that were within two of the target range as meeting the constraint because it was hard for students to count correctly if they did not use grouping or arrays in their design.

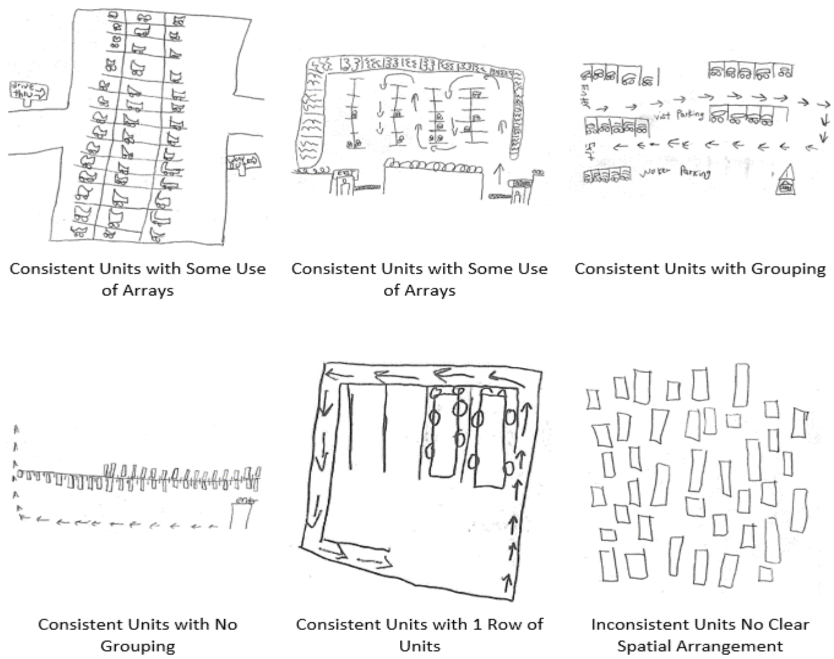

Figure 2. Parking Lot Designs (from day 2)

Table 3 shows the strategies students used in creating their dog parks in lesson 3.

Table 3: Strategies Used in Designing Dog Park

Strategies	Number of Students (n=22)
Consistent Units with Some Use of Arrays	3
Consistent Units with Grouping	7
Consistent Units with no Grouping	7
Consistent Units with 1 Row of Units	0
Inconsistent Units No Clear Spatial Arrangement	1
No Use of Units Shown	4

Only 10 students made use of grouping and arrays to create their dog parks. Most students (n = 17) made use of consistent units in their designs. There were two constraints we considered for the design of the park. First, they must provide an area suitable in size for 24 dogs needing to move around and exercise. Second, they must provide a rectangular park shape overall. We found students sometimes met both constraints, but still they were not successful in the total design project. For example, some students created a rectangular area that had room for more than 24 dogs even though they showed the space for 24 dogs. Other students accounted for the space for 24 dogs but appointed another rectangle to be the actual dog park (See Table 4). Figure 3 shows some examples of dog park designs.

Table 4: Met Dog Park Constraints

Constraints	Number of Students (n=22)
Area of Park Design for 24 Dogs	6
Dog Park Rectangle	4
Both Constraints Met	6
Neither Constraint Met	6

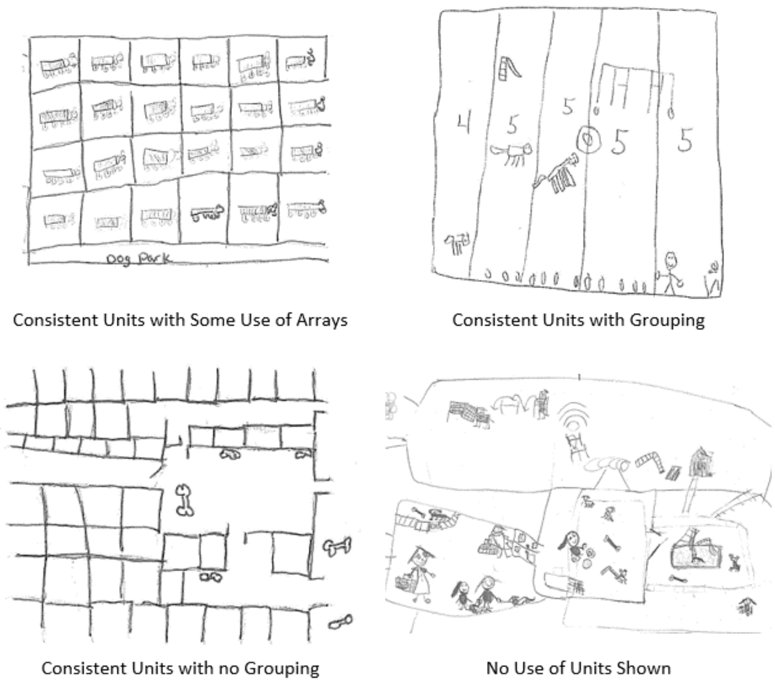

Figure 3. Dog Park Designs (from day 3)

Conclusion

Given the brief span of the intervention we conducted, we were not expecting students to move on to a new level of the learning trajectory (LT) for area measurement strategies. Rather, we used the LT levels as a rubric to find a suitable instructional level given students' exhibited knowledge of area measurement. Our findings with these design-focused tasks suggest students were creating designs and engaging with area measurement tasks that involved multiplication schemes in productive ways which is in keeping with AURR levels. This finding suggests design-centered tasks of this type offer ways of supporting student thinking and of observing their reasoning at these particular levels of a LT for area. This may provide a way of improving the instructional task descriptions as the LT is modified to broaden its impact on a wider range of students in various contexts and communities.

Furthermore, the design process of instruction appears helpful in focusing students' attention on the meaningful association among arrays, multiplication operations with number, and the measure of rectangular shapes. By engaging contexts that fit with our observations about the students' own experiences, we apparently gained access to familiar stories from their daily routines and community-based language for spatial quantity. The teacher was pleased to note that several students who typically did not engage in mathematics stayed engaged with the tasks for as long as they did. More work is needed to find what motivated this level of investment in the tasks.

Dealing with design constraints had mixed results from our vantage point; some students did not address any of the constraints, although other students successfully addressed one or more constraints. For the second lesson, 14 of 22 students met the design constraint for the parking lot. For the third lesson, 16 of 22 students met at least one of two constraints requested.

Nevertheless, students in Grade 3 demonstrated the capability to address design constraints related to measurement and space. The design emphasis, with the integration of measurement, multiplication schemes, and arrays as tools appears to be a viable way to adapt learning trajectory-based activities for area measurement. The lesson outcomes indicate promise that students in Grade 3 can engage in design activities with constraints related to multiplicative reasoning using skip-counting and grouping schemes. The interaction among these schemes may have prompted students to engage in the quantitative reasoning by access to their knowledge of such contexts as scanning to find whether a parking lot is empty, partly filled, or full. We believe the prominence and meaningfulness of the context provided a way for the mathematics of area measurement to be addressed as an integrated part of instruction on multiplication and arrays. This is consistent with work in statistics education showing the importance of linking context knowledge to the statistical schemes for organizing and reporting on data in such a context (Langrall, Nisbet, & Mooney, 2006).

We believe further design cycles may need to draw out a more comprehensive analysis of the multiplication processes and the arrays as tools for measuring the capacity of a parking lot to hold cars. We plan further work with the same students to have them redesign a dog park to meet constraints related to the area measurement and to the shape of the region (by requiring a rectangle). We also expect to include further ways to prompt students to check their own design by using a grouping scheme for collections of units. This could focus them more on iterating squares to fill space, and link to arrays and measuring area. The process of testing, designing, retesting and redesigning are vital STEM skills for students to develop (https://stem.getintoenergy.com/stem-skills-list/). The redesign process is important as we learn to extend the instructional tasks found in learning trajectories (e.g., the area LT) to different communities.

Ideally, teachers will use similar design-based tasks to adapt and work with their students in different community contexts. The principles of designing, measuring and describing, taken from analyses across a wide range of culture and communities by Bishop (1988) may productively inform both teachers and researchers who want to adapt learning trajectories for other content areas. Our findings suggest that designing, describing and measuring may be productive ways of engaging students as young as grade 3 in substantive mathematical projects. This may support them as they learn the structural advantages of noticing or setting up arrays to support area measurements and multiplication operations.

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